4-Manifolds and Kirby calculus

Webpage: https://www.mathematik.hu-berlin.de/~kegemarc/SS21Kirby.html
OneNote link: https://1drv.ms/u/s!AjhcHi01JrMRgVb9QfZqwGtEXDJW

1. OVERVIEW:

smooth n-MANIFOLD M^n $\approx$ top space, locally homeom to $\mathbb{R}^n$ (+smooth)

often: CLOSED (i.e. compact & without boundary)

orientable, connected

$\boxed{\text{POINCARÉ CONJECTURE}}$ (~1903)

M^n closed with $M \simeq S^n$

HOMOTOPY EQUIVALENT

$\Downarrow$?

$M \overset{C^0}{\cong} S^n$

HOMEOMORPHISM

$\boxed{\text{YES!}}$

$n=1,2$ (Exercise sheet 1)

$n=3$ PERELMAN 2003

$n=4$ FREEDMAN 1981

$n \geq 5$ SMALE 1960

(Chapter 10)

(HANDLE DECOMPOSITIONS)

$\Downarrow$?

$M \overset{C^\infty}{\cong} S^n$

DIFFEOMORPHISM

$n=1,2,3$ YES

$n=7$ NO! MILNOR 1956

$n \geq 5$ well-understood

(e.g. finite)

$\boxed{n=4}$ $\boxed{?}$

THM (DONALDSON ~1980)

On $\mathbb{R}^n$ $(n \neq 4)$ $\exists!$ smooth str.

On $\mathbb{R}^4$ $\exists$ uncountably many different smooth str. (Chapter 11)

THM (Chapter 11)

$\exists$ compact smooth 4-mfds M_1, M_2 s.t. $M_1 \overset{C^0}{\cong} M_2$ but $M_1 \overset{C^\infty}{\ncong} M_2$

THM (WALL) (Chapter 7)

Let M_1, M_2 be closed 4-mfd with $\pi_1(M_i) = 1$

If $M_1 \overset{C^0}{\cong} M_2 \Rightarrow \exists k \in \mathbb{N}_0 : M_1 \#_k S^2 \times S^2 \overset{C^\infty}{\cong} M_2 \#_k S^2 \times S^2$

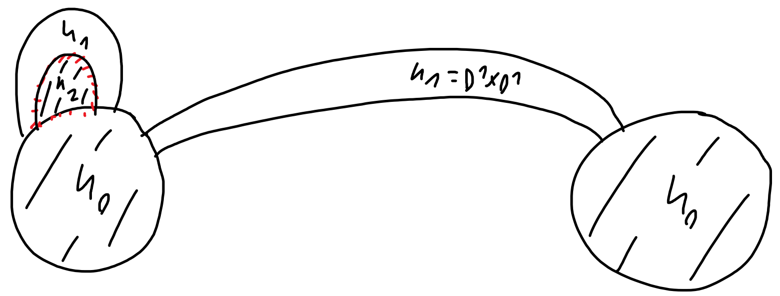
HANDLE:

$$\underset{\substack{\uparrow \\ \text{INDEX}}}{h_k} = D^k \times D^{n-k} \overset{C^0}{\cong} D^n$$

HANDLE DECOMPOSITION:

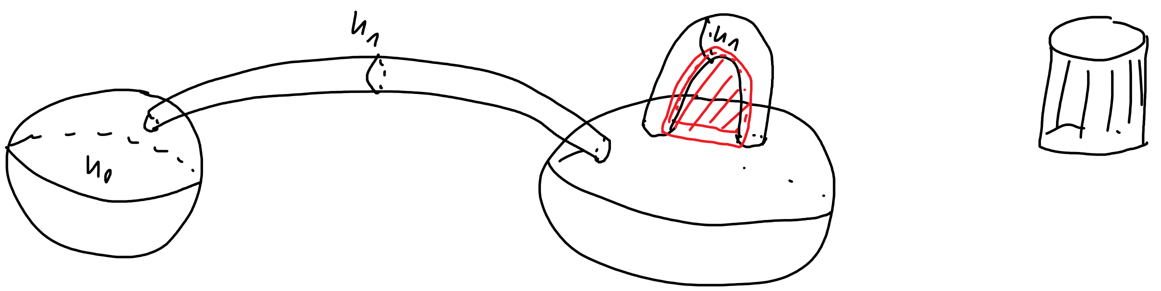
h_k is ATTACHED along $\partial D^k \times D^{n-k} = \underset{\substack{\uparrow \\ \text{ATTACHING SPHERE}}}{S^{k-1}} \times D^{n-k}$

$\boxed{n=2}$



$h_2 = D^2 \times D^1$

$\boxed{n=3}$



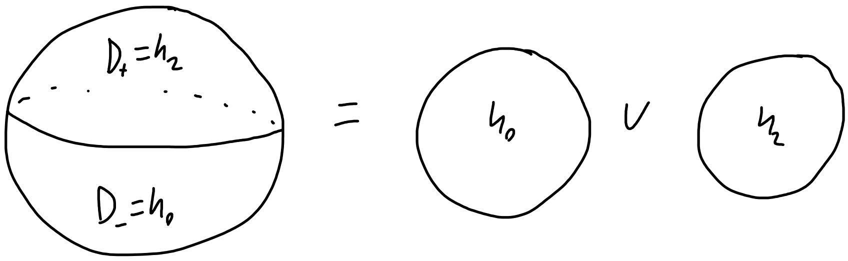
THM (SMALE) (Chapter 2)

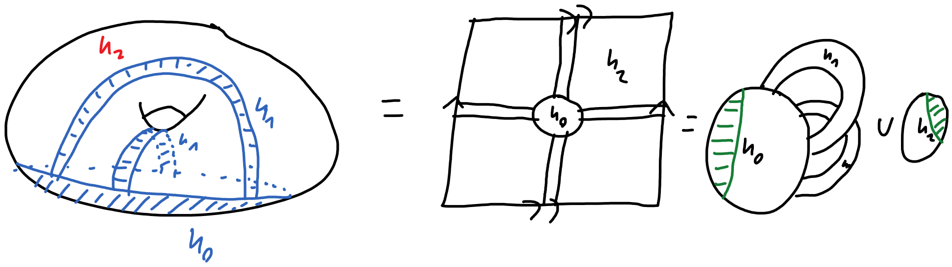
'Any' n-mfd admits a handle decomposition with

a unique 0-handle &

" n-handle.

Examples:

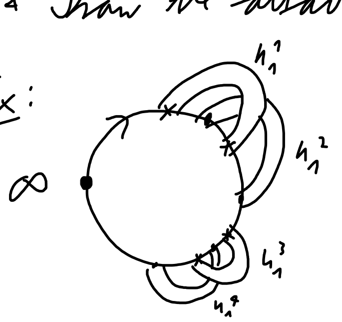
(1) $S^2 =$ 

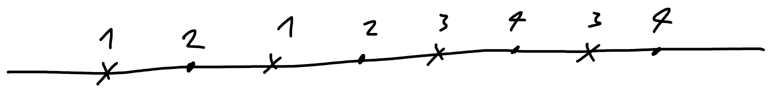
(2) $T^2 =$ 

(3) $\#_g T^2 = \Sigma_g =$ 

DIM 2: "KIRBY CALCULUS ON SURFACES"

* Let F^2 be a surface (= 2-mfd) with handle decomp $h_0 \cup h_1^1 \cup \dots \cup h_1^k \cup h_2$
* Consider $\partial h_0 = \partial D^2 = S^1 = \mathbb{R} \cup \{\infty\}$
* Draw the attaching spheres of the 1-handles h_1^i on $\mathbb{R} \subset \partial h_0$

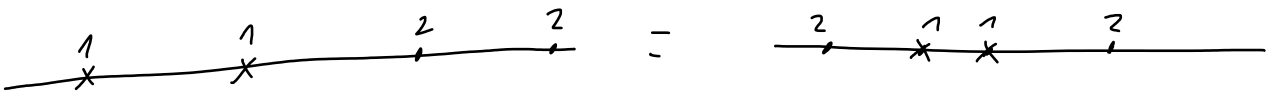
Ex: 

= 

$\mathbb{R} \subset \partial h_0$

"KIRBY DIAGRAM OF A SURFACE"

HANDLE SLIDE



<u>Exercise sheet 1</u>: Classify surfaces via handle calculus

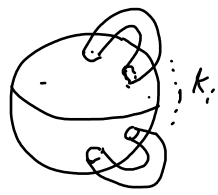
<u>THM</u>: $\forall F^2 \; \exists! \, k \in \mathbb{N}_0 : F \overset{C^\infty}{\cong} \#_k T^2$

DIM = 3: HEEGAARD DIAGRAMS:

Let M^3 be a 3-mfd with handle decomp:

$$\underbrace{h_0 \cup h_1^1 \cup \dots \cup h_1^{k_1}}_{H_1} \cup \underbrace{h_2^1 \cup \dots \cup h_2^{k_2} \cup h_3}_{H_2}$$

<u>CLAIM</u>: $k_1 = k_2$

$H_i =$  $= 3$-dim 1-handle bodies $= \natural_{k_i} S^1 \times D^2$

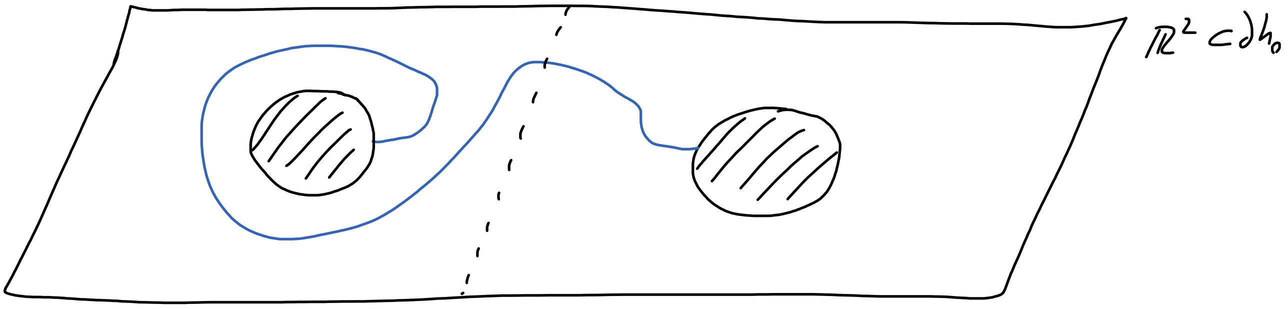
$$\left[\Sigma_{k_1} = \partial H_1 = \partial H_2 = \Sigma_{k_2} \Leftrightarrow k_1 = k_2 \right]$$

$$\left(D^k \times D^{n-k} = D^{n-k} \times D^k \right)$$

$M^3 = H_1 \cup H_2$ is called <u>HEEGAARD SPLITTING</u>

* Consider $\partial h_0 = \partial D^3 = S^2 = \mathbb{R}^2 \cup \{\infty\}$

* Draw attaching regions of 1-handles h_1^i & 2-handles h_2^i in $\mathbb{R}^2 \subset \partial h_0$

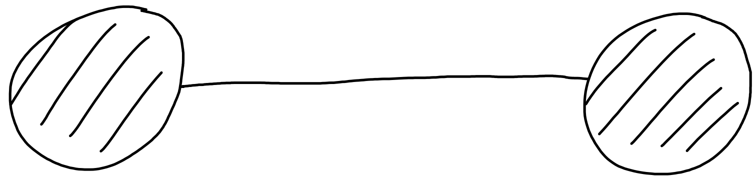


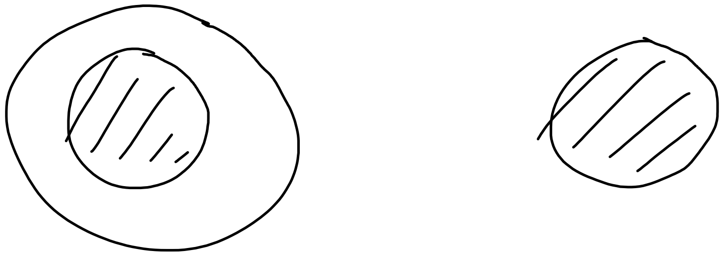
identify the two sides via $(x,y) \longmapsto (-x,y)$

* Attaching spheres of 2-handles:

$$S^1 = \partial D^2 \times \{0\} \subset \partial \left(h_0 \cup h_1^1 \cup \dots \cup h_1^k\right)$$

Ex:

(1)  $= S^3$

(2)  $= S^1 \times S^2$

DIM=4 : KIRBY DIAGRAMS :

Let W^4 be a 4-mfd with handl. decomp

$$W = h_0 \cup h_1^1 \cup \dots \cup h_1^{k_1} \cup h_2^1 \cup \dots \cup h_2^{k_2} \cup h_3^1 \cup \dots \cup h_3^{k_3} \cup h_4$$

* Consider $\partial h_0 = \partial D^4 = S^3 = \mathbb{R}^3 \cup \{\infty\}$
* Draw attaching regions in $\mathbb{R}^3 \subset \partial h_0$

Attaching region of 1-handle: $S^0 \times D^3 = D^3 \sqcup D^3 \subset \mathbb{R}^3$

Attaching 1-handle $\Leftrightarrow$ identifying $D^3 \sqcup D^3$ via a reflection

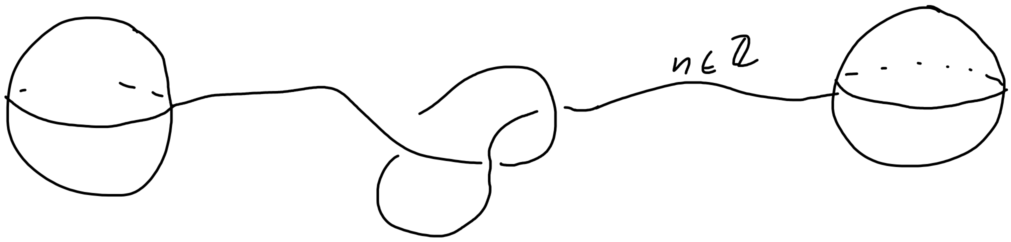
1-handlebody $= W_1 = h_0 \cup h_1^1 \cup \dots \cup h_1^{k_1} = \natural_{k_1} S^1 \times D^3$

$W_k :=$ handles of index $\leq k$

Attaching sphere of 2-handle :

attaching map $\varphi : \partial D^2 \times D^2 \hookrightarrow \partial W_1$

$$K := \varphi(\partial D^2 \times \{0\}) \subset \partial W_1 \quad \text{a knot}$$



K together with a <u>FRAMING</u> determines φ

$$h_3^1 \cup \dots \cup h_3^{k_3} \cup h_4 = \natural_{k_3} S^1 \times D^3$$

$$\Rightarrow \partial W_2 = \partial\left(\natural_{k_3} S^1 \times D^3\right) = \#_{k_3} S^1 \times S^2$$

<u>LAUDENBACH - POÉNARU (1972)</u>

$$\forall f : \#_m S^1 \times S^2 \xrightarrow{\overset{C^\infty}{\cong}} \#_m S^1 \times S^2$$

$$\exists F : \natural_m S^1 \times D^3 \xrightarrow{\overset{C^\infty}{\cong}} \natural_m S^1 \times D^3 \qquad F|_\partial = f$$

$\Rightarrow$ W is determined by W_2

$$\begin{tikzcd}
W \arrow[d, dotted] & = & W_2 \arrow[d, "\mathrm{id}"] & \cup_{\varphi_2} & \natural_m S^1 \times D^3 \arrow[d, "\text{extension } \varphi_1 \circ \varphi_2^{-1}"] \\
W' & = & W_2 & \cup_{\varphi_1} & \natural_m S^1 \times D^3
\end{tikzcd}$$

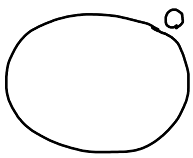
Examples:

(1) empty diagram $\emptyset$

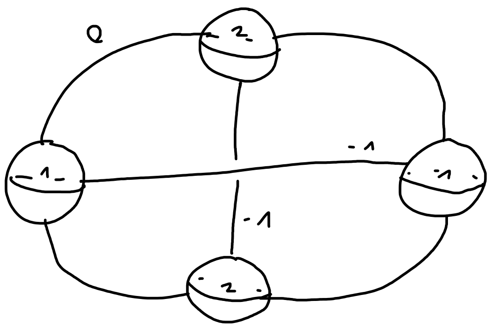
$W_2 = D^4$ $\qquad$ $\partial W_2 = S^3$ $\qquad$ $W = S^4$

(2) 

$W_2 = S^1 \times D^3$ $\qquad$ $\partial W_2 = S^1 \times S^2$ $\qquad$ $W = S^1 \times S^3$

(3) 

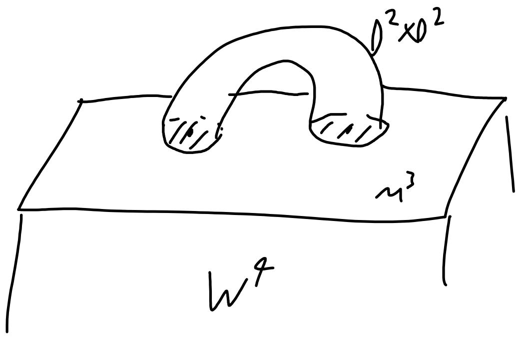
$W_2 = D^2 \times S^2$ $\qquad$ $\partial W_2 = S^1 \times S^2$ $\qquad$ $W = S^4$

(4) 

$= \ ?$

Remark: A Kirby diagram of W_2 is a surgery diagram of ∂W_2

Let W^4 with $\partial W = M^3$



Attaching a handle to W

$\parallel$

performing a surgery on ∂W

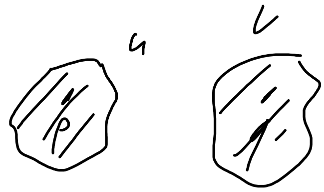
$$M' = M \setminus (S^1 \times D^2) \cup_\varphi (D^2 \times S^1)$$

2. MANIFOLDS & HANDLE DECOMPOSITIONS

2.1. MANIFOLDS

Def: M^n is a (TOP) MANIFOLD OF DIMENSION n :$(\Leftrightarrow)$

(1) M is a top HAUSDORFF space



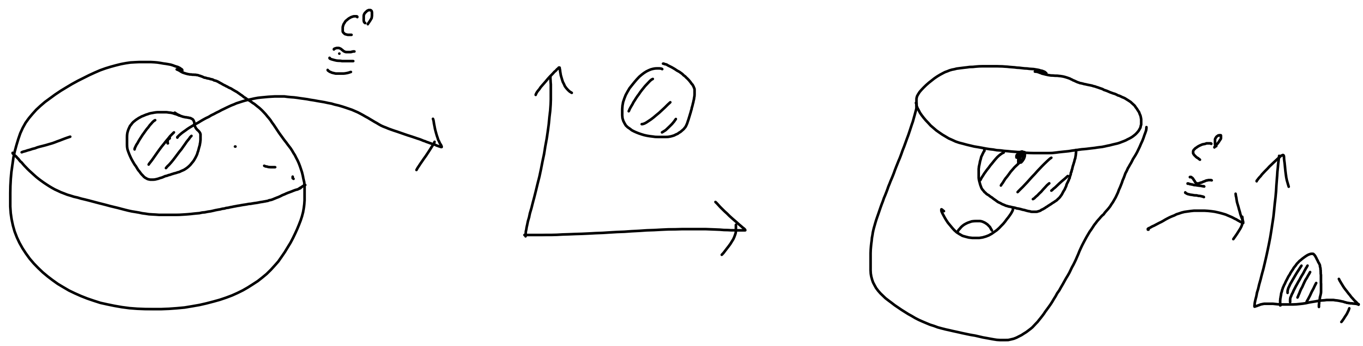
(2) M has a countable basis

(3) $\forall p \in M \quad \exists\, p \in U \subset M$ open &

$$\exists\ \varphi : U \xrightarrow{\cong_{C^0}} V \subset \mathbb{R}^n \text{ open}$$

φ = CHART $\qquad \varphi^{-1}$ = PARAMETRIZATION

(Replace $\mathbb{R}^n$ by $\mathbb{R}^n_+ := \{x_n \geq 0\}$ $\longrightarrow$ MFD WITH BOUNDARY)

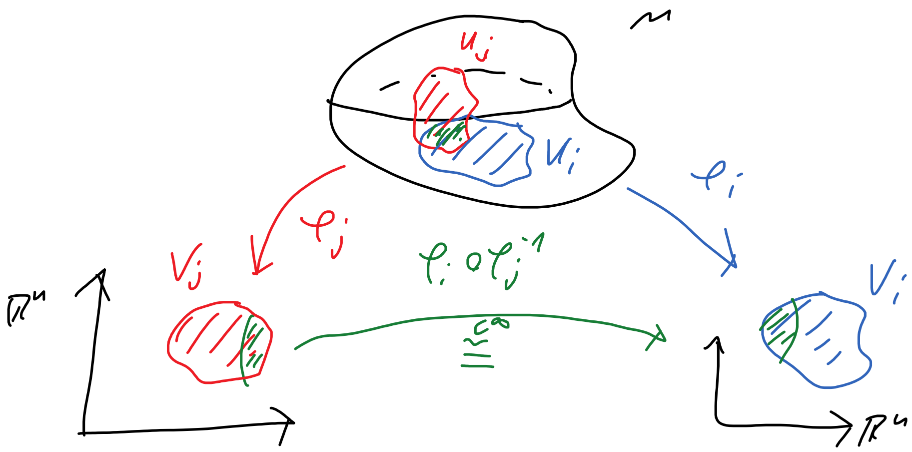


* An ATLAS $\mathcal{A}$ of M is a family of charts $\{(U_i, \varphi_i)\}_{i\in I}$

s.t. $\quad M = \bigcup_{i \in I} U_i$

* $\{U_i, \varphi_i\}$ & $\{U_j, \varphi_j\}$ are COMPATIBLE :$\Leftrightarrow$

$$\varphi_i \circ \varphi_j^{-1} : \quad \underset{\mathbb{R}^n}{\varphi_j(U_i \cap U_j)} \xrightarrow[\cong]{C^\infty} \underset{\mathbb{R}^n}{\varphi_i(U_i \cap U_j)}$$



* $\mathcal{A}_1$ & $\mathcal{A}_2$ are EQUIVALENT :$\Leftrightarrow$ all charts in $\mathcal{A}_1 \cup \mathcal{A}_2$ are compatible

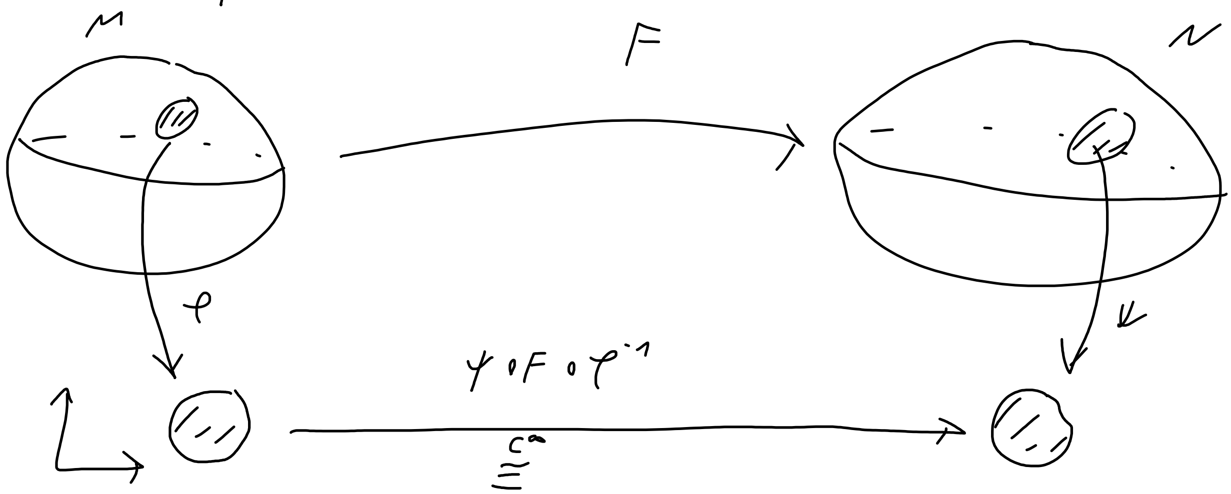
* The equivalence class of an atlas (in which all charts are compatible) is called SMOOTH STRUCTURE.

* $F : M \longrightarrow N$ is called DIFFEOMORPHISM :$\Leftrightarrow$

(1) F is a homeomorphism

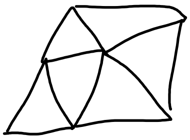
(2) $\forall$ charts (U, φ) of M $\quad \forall$ charts (V, ψ) of N:

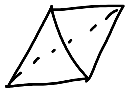
$$\psi \circ F \circ \varphi^{-1} \in C^\infty$$



Remark:

* If we replace C^∞ by PL we get the class of PL mfds
* PL mfds have triangulations



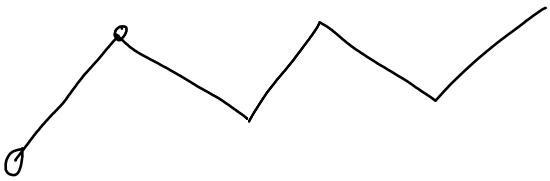


We have:

* $$\mathrm{DIFF} \subset \mathrm{PL} \subset \mathrm{TOP}$$

(WHITEHEAD)

* i.g. $\mathrm{DIFF} \neq \mathrm{PL} \neq \mathrm{TOP}$
* $n = 1, 2, 3$: $\mathrm{TOP} = \mathrm{PL} = \mathrm{DIFF}$ (MOISE 1953)
* $n = 4$: $\mathrm{TOP} \neq \mathrm{PL} = \mathrm{DIFF}$



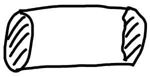
2.2. HANDLE DECOMPOSITIONS

Def: An n-dim <u>K-HANDLE</u> (INDEX) h_K is copy of $D^K \times D^{n-K}$ <u>ATTACHED</u> to a smooth mfd M^n via an embedding

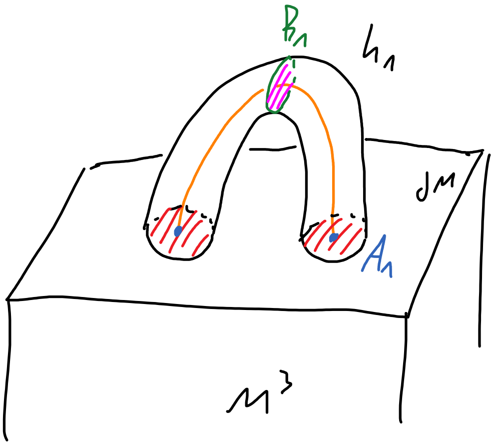
$$\varphi : \partial D^K \times D^{n-K} \hookrightarrow \partial M$$

Ex: $\boxed{n=3 \quad k=1}$ $\boxed{n=3 \quad k=2}$

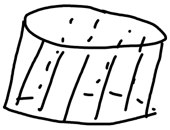
$h_1 = D^1 \times D^2$



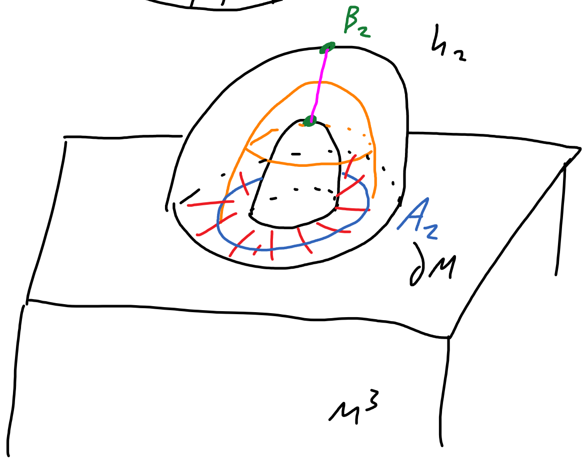
$\partial D^1 \times D^2$



$h_2 = D^2 \times D^1$



$\partial D^2 \times D^1$



ATTACHING REGION $= \partial D^k \times D^{n-k} \equiv \varphi(\partial D^k \times D^{n-k})$

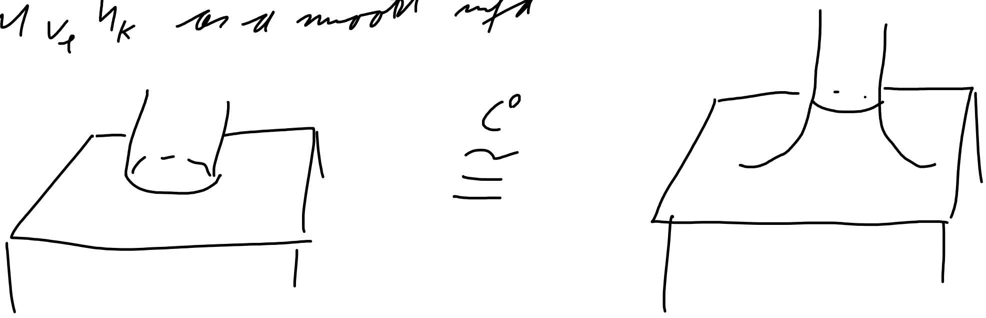
ATTACHING SPHERE $A_k = \partial D^k \times \{0\} = S^{k-1}$

BELT SPHERE $B_k = \{0\} \times \partial D^{n-k}$

CORE $= D^k \times \{0\}$

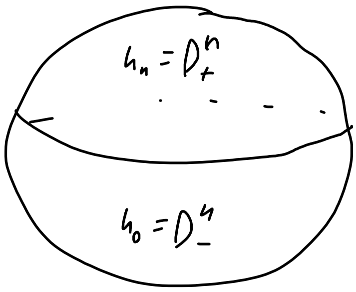
CO-CORE $= \{0\} \times D^{n-k}$

Remark: We see $M \cup_\varphi h_k$ as a smooth mfd

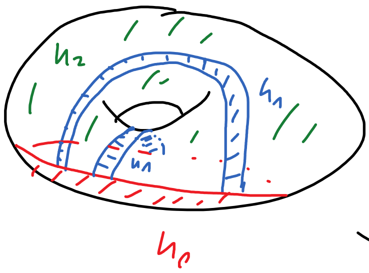


Ex:

(1) $S^n = D^n_+ \cup D^n_- = h_0 \cup h_n$



(2) $T^2 =$



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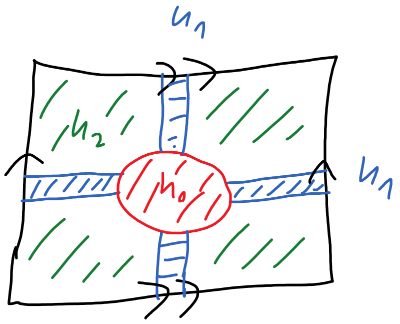

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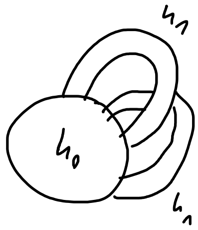

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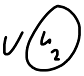
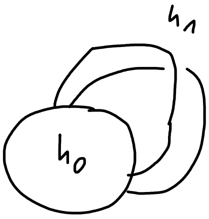
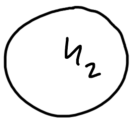


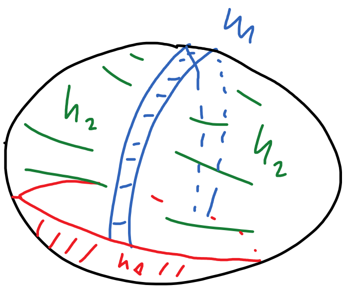
(3) $\mathbb{RP}^2 =$



$\cup$



(4) $S^2 =$



Lemma 1: $e_i : \partial D^k \times D^{n-k} \hookrightarrow \partial M$ for $i = 1,2$

e_1 isotopic to e_2 $\Rightarrow$ $M \cup_{e_1} h_k \overset{C^\infty}{\cong} M \cup_{e_2} h_k$ $\lrcorner$

Proof: $\ulcorner$ ISOTOPY EXTENSION THM:

M, N compact & $h : I \times N \longrightarrow M$ isotopy

$\Rightarrow$ $\exists\ H : I \times M \longrightarrow M$ s.t. $\Big\}$ AMBIENT ISOTOPY

* $H_0 = id_M$
* $H_t =$ a diffeo $\forall t \in I$
* $h_t = H_t \circ h_0$ $\lrcorner$

$\llcorner$

$$\begin{tikzcd}
M_1 \arrow[d, dotted, "\cong"'] & = & M \arrow[d, "H_1\ \cong"'] & + & h_k \arrow[d, "\cong\ id"] & /\sim \\
 & & e_1(p) & \circlearrowleft & p \arrow[ll, mapsto] & \\
 & & e_2(p) & & p \arrow[ll, mapsto] & \\
M_2 & = & M & + & h_k & /\sim
\end{tikzcd}$$

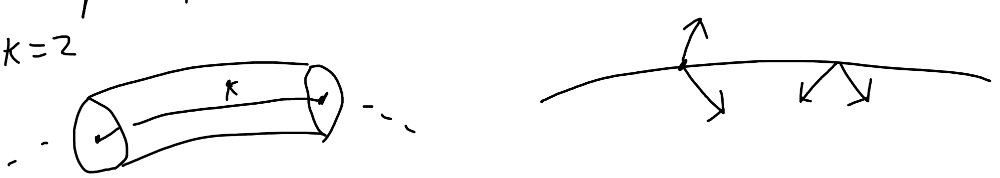
$\blacksquare$

Remark:

The isotopy class of $e : \partial D^k \times D^{n-k} \hookrightarrow \partial M$ is determined by $e|_{\dots} = e_0 : \partial D^k \times \{0\} = S^{k-1} \hookrightarrow \partial M$ together with a FRAMING of $e_0(S^{k-1}) =: K \subset \partial M$, i.e.

a map $K \longrightarrow GL_{n-k}(\mathbb{R})$

$n = 4$, $k = 2$



Thm 2 (SMALE, 1960)

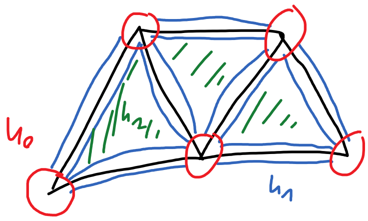
$\forall$ smooth, compact mfd M $\exists$ a handle decomp of M

Proof idea:

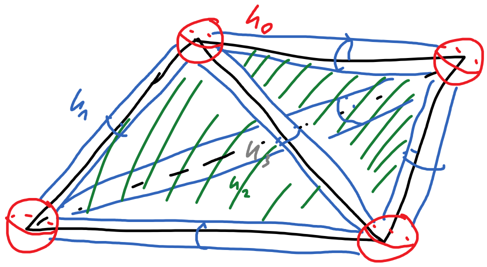
(1) PL:

* Let T be a triangulation of M

$n = 2$



$n = 3$

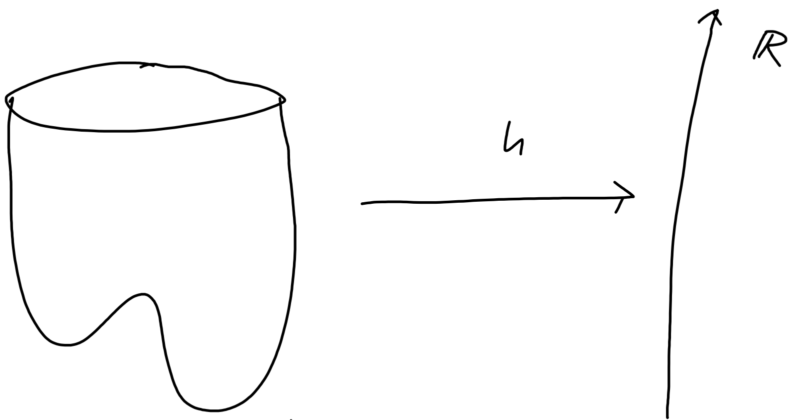


regular nbhd of k-simplex $\cong$ k-handle

(2) C^∞: **MORSE-THEORY:**

Choose an embedding $M \subset \mathbb{R}^n$ (WHITNEY)

Consider $h: M \longrightarrow \mathbb{R}$



h MORSE $:\Leftrightarrow$ $\forall$ CRITICAL POINT $p \in M$

(i.e. $\partial_p h = 0$):

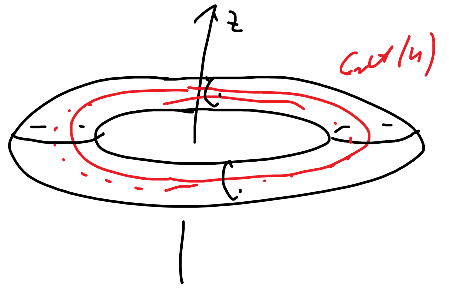
$\Rightarrow \det(H_p h) \neq 0$

$\Leftrightarrow$ $\forall p \in \mathrm{Crit}(h)$ $\exists$ coord $(x_1, \dots, x_n)$ s.t.

$$h: (x_1, \dots, x_n) \longmapsto -\sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2$$

$\Leftrightarrow$ h generic

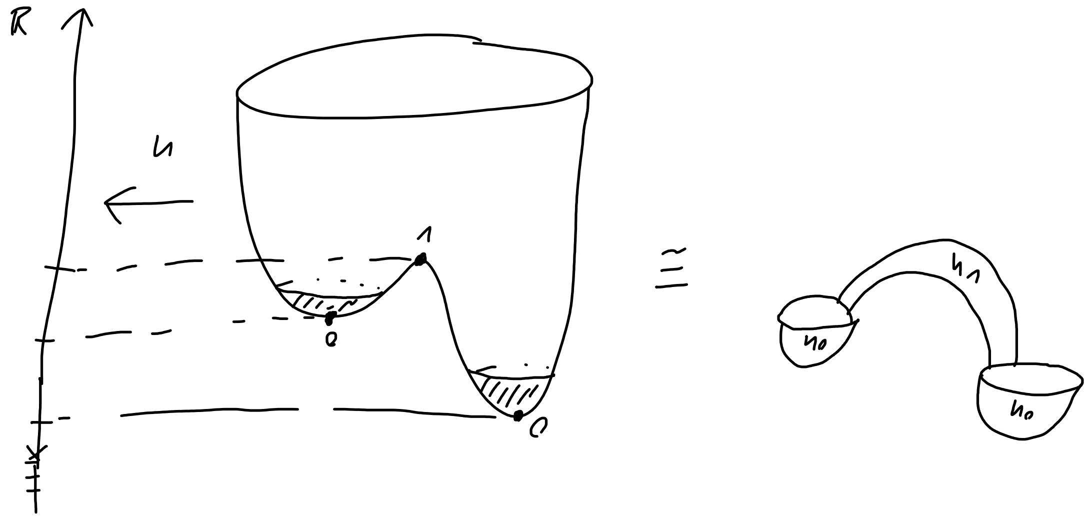
$\Rightarrow \ \forall M \ \exists h : M \longrightarrow \mathbb{R}$ morse



$(x,y,z) \mapsto z$

$\longrightarrow \mathbb{R}$ IS NOT MORSE

Observation: Crit points of h $\longleftrightarrow$ k-handle



$\blacksquare$

Lemma 3 for $\ell \leq k$:

$$(M \cup h_k) \cup h_\ell \overset{C^\infty}{\cong} (M \cup h_\ell) \cup h_k$$

Proof sketch:

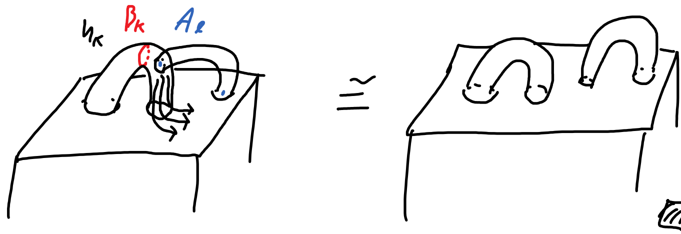
Let $A_\ell = S^{\ell-1} \subset \partial(M \cup h_k)$ the attaching sphere of h_ℓ

& $B_k = S^{n-k-1} \subset \partial(M \cup h_k)$ the belt sphere of h_k

$$\dim(A_\ell) + \dim(B_k) = \ell - 1 + n - k - 1 \overset{(\ell \leq k)}{<} n - 1 = \dim(\partial(M \cup h_k))$$

transversality thm

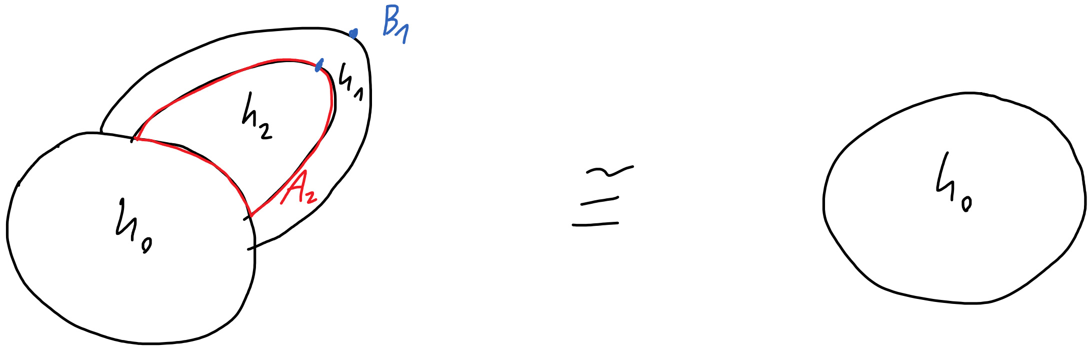
$\Rightarrow A_\ell \pitchfork B_k = \emptyset$

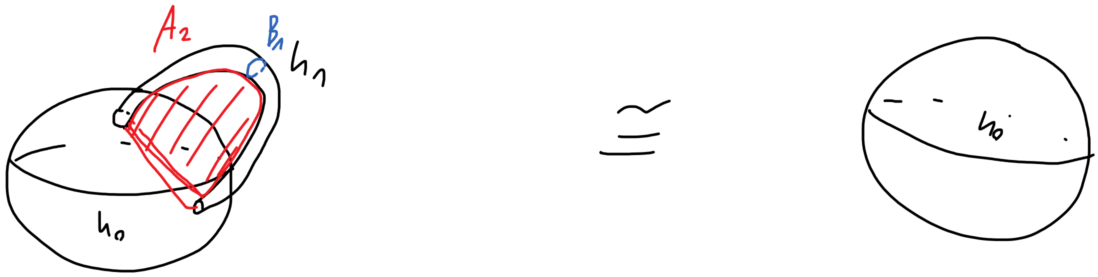


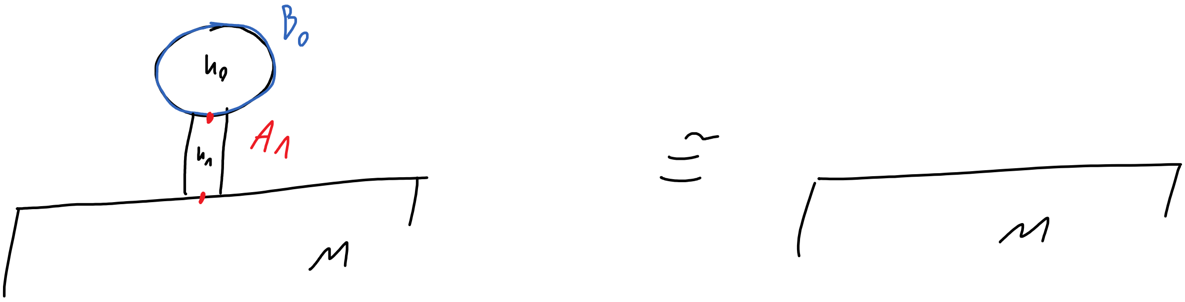
$\blacksquare$

HANDLE CANCELLATION

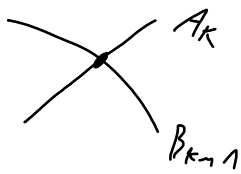
Ex:







Lemma 4: If $A_k \pitchfork B_{k-1} = \{pt\}$



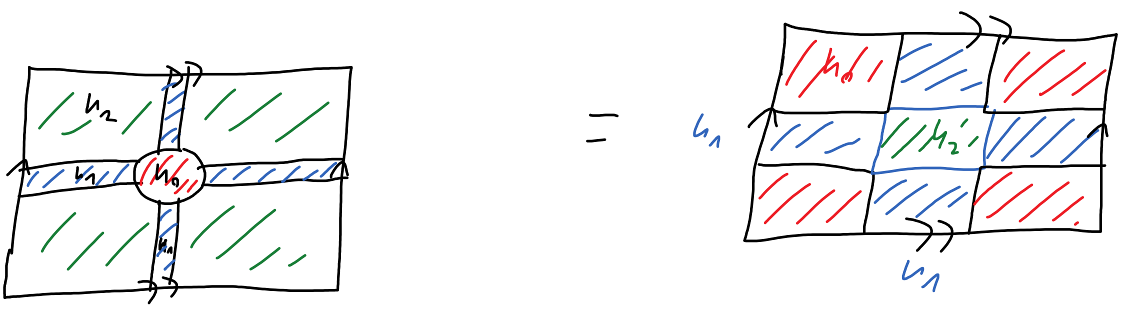
$$\Rightarrow (M \cup h_{k-1}) \cup h_k \cong M$$

∎

DUAL HANDLE DECOMPOSITION:

Observe: k-handle $h_k = D^k \times D^{n-k} = D^{n-k} \times D^k = (n-k)$-handle h_{n-k}

cocore of h_k = core of h_{n-k}



__Lemma 5:__

M^n connected closed

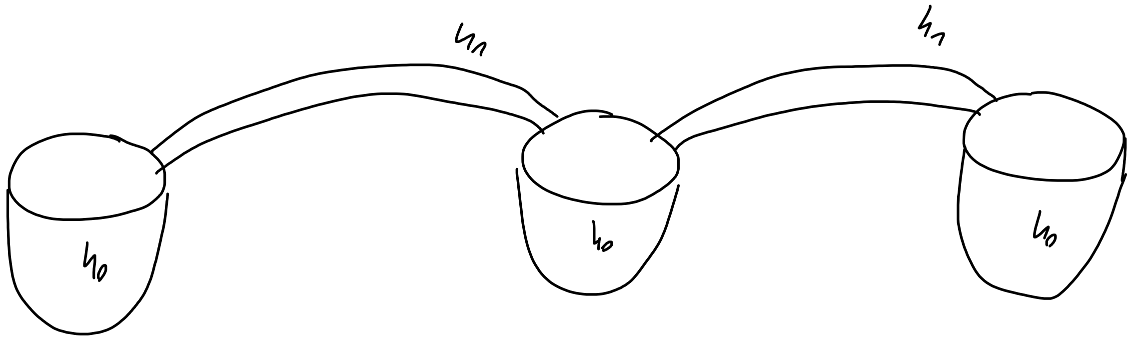
$\Rightarrow$ $\exists$ handle decomp of M with

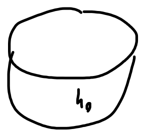
- exactly one 0-handle
- " n-handle

__Proof:__

* M closed $\Rightarrow$ every handle decomp. has at least one 0-handle

* M connected $\Rightarrow$ h_0^i are connected by 1-handles



CANCELLATION $\cong$ 

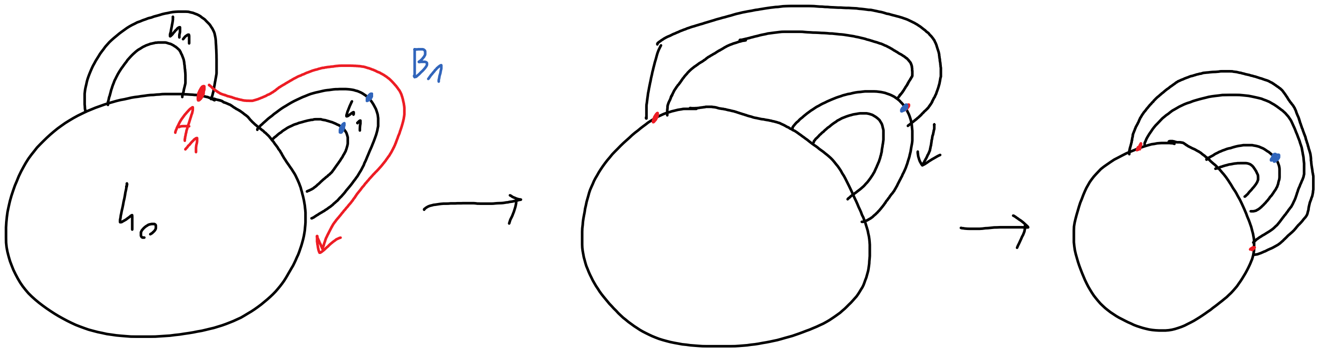
$\Rightarrow$ $\exists!$ 0-handle

* dual handle decomp $\Rightarrow$ $\exists!$ n-handle

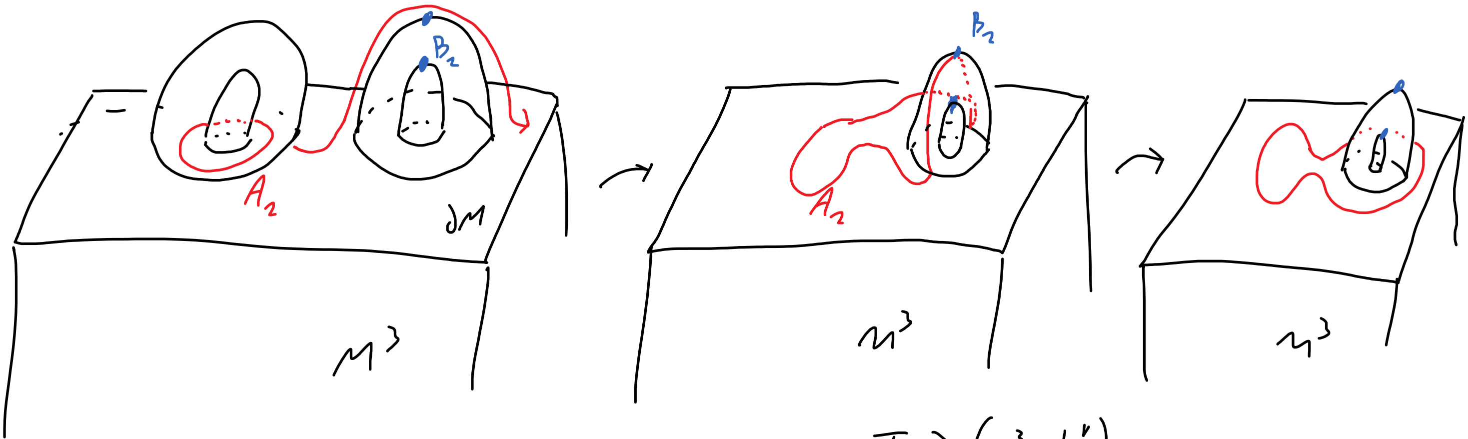
∎

HANDLE SLIDES:

Ex: $n=2 \quad k=1$



$n=3 \quad k=2$



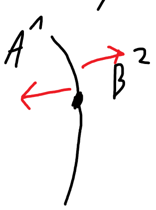
$$A \pitchfork B \Leftrightarrow \forall p \in A \cap B: \quad T_pA + T_pB = T_p \partial(M^3 \cup h_2^1)$$

Let h_k^1, h_k^2, $0<k<n$ be two k-handles attached to ∂M.

A <u>HANDLE SLIDE</u> of h^1 over h^2 is the isotopy of A^1 in $\partial(M \cup h^2)$ through the B^2.

$\ulcorner$

$$\dim(A^1) + \dim(B^2) = k-1 + n-k-1 = n-2 = \dim(\partial(M \cup h^2)) - 1$$



$\llcorner \qquad \lrcorner$

THM 6 (CERF, 1970)

* Two handle decompositions (ordered by increasing index) of a compact mfd M are related by finitely many handle slides and introducing/removing cancelling pairs.

* If the handle decomp. has a unique 0- & n-handle, we do <u>NOT</u> need to introduce cancelling pairs 0/1 & $n-1$/n handles.

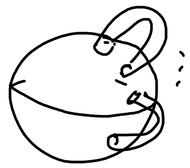
<u>Proof:</u>

$$
\begin{tikzcd}
\text{handle decomp } 1 \arrow[d, leftrightarrow] & & \text{handle decomp } 2 \arrow[d, leftrightarrow] \\
\text{Morse fct } h_1 : M \to \mathbb{R} \arrow[rr, squiggly, "h_t : M \to \mathbb{R}"'] & & h_2 : M \to \mathbb{R} \quad t \in [1,2]
\end{tikzcd}
$$

a generic path of Morse fct connecting h_1 & h_2 $\blacksquare$

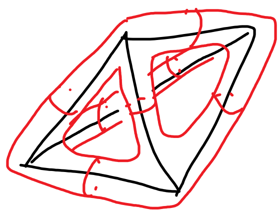
3. DIM 3: HEEGAARD SPLITTINGS:

GOAL: $\forall M^3 = H_1 \cup H_2$

$H_i =$ 

IDEA: Let T be a triang. of M

H_1 = regl. nbhd of vertices & edges



$H_2 = M \setminus H_1$

PROBLEM: This is WRONG if M is NOT or.

3.1. HEEGAARD SPLITTINGS

Let M^3 be a connected, closed, orientable 3-mfd with a handle decomp

$$M = \underbrace{h_0 \cup h_1^1 \cup \dots \cup h_1^{g_1}}_{=: H_1} \cup \underbrace{h_2^1 \cup \dots \cup h_2^{g_2} \cup h_3}_{=: H_2}$$

Def: A smooth mfd M^n is called ORIENTABLE :$\Leftrightarrow$

$\exists$ Atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ of M s.t.

$$\forall p \in M \ \forall i,j : \det(\partial_p(\varphi_j \circ \varphi_i^{-1})) > 0$$

$\boxed{\text{„} \nexists \text{ loop in } M \text{ interchanging left and right"}}$

Lemma 1:

M^n be a smooth, orientable, compact with a handle decomp; $n \geq 3$

$$\Rightarrow M_1 := \{0\text{-handle}\} \cup \{1\text{-handles}\} \overset{C^\infty}{\cong} \natural_g \, S^1 \times D^{n-1}$$

↑ 1-HANDLEBODY OF GENUS g

NOTATION: $M_k := \{\text{handles of index} \leq k\}$

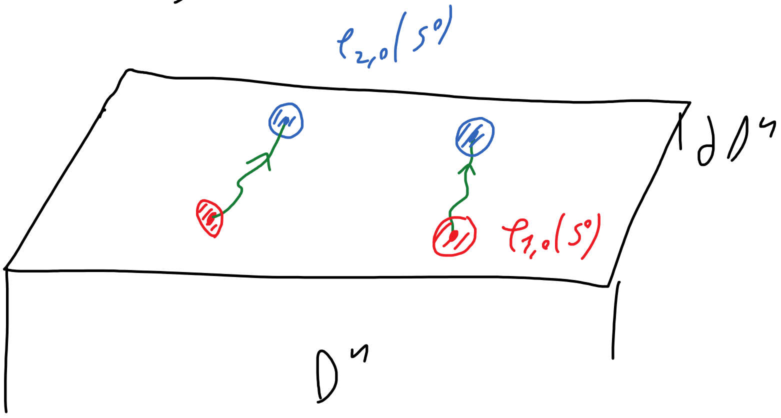
Proof: We show: $\forall\, e_1, e_2 : \partial D^1 \times D^{n-1} \hookrightarrow \partial D^n$

are isotopic

$$\left(\text{Idea} : \ L\, 2.1 \Rightarrow D^n \cup_{e_1} h_1 \overset{C^\infty}{\cong} D^n \cup_{e_2} h_1 \right)$$

* Two embeddings $e_{i,0} : \underset{\substack{\parallel \\ S^0}}{\partial D^1} \times \{0\} \hookrightarrow \partial D^n$

are isotopic



* Framings of $K := \varphi_0(\partial D^1 \times \{0\}) \subset \partial D^n$
are homotopy classes of maps

$$K = S^0 \longrightarrow GL_{n-1}(\mathbb{R})$$

$\Rightarrow$ $\{$framings of $K\}$ $= \pi_0(GL_{n-1}(\mathbb{R}))$ = connected comp of $GL_{n-1}(\mathbb{R})$
$= \mathbb{Z}_2$

M orientable $\Rightarrow$ $\exists!$ framing of K along which to attach a 1-handle. $\square$

Lemma 2: H_1 & H_2 are 1-handlebodies of the same genus.

Proof: * $\angle 1$ $\Rightarrow$ H_1 is a 1-handlebody

$\Rightarrow$ $\partial H_1 = \Sigma_{g_1}$

* dual handle decomp & $\angle 1$ $\Rightarrow$ H_2 is a 1-handlebody

$\Rightarrow$ $\partial H_2 = \Sigma_{g_2}$

* $\Sigma_{g_1} = \partial H_1 = \partial H_2 = \Sigma_{g_2}$

$\Leftrightarrow$ $g_1 = g_2$ $\square$

Def: A decomposition of M^3 into two 1-handlebodies of the same genus: $M = H_1 \cup H_2$

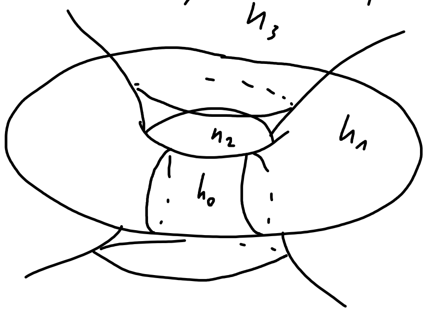
is called HEEGAARD SPLITTING.

Corollary 3: $\forall$ closed, orientable 3-mfd M $\exists$ Heegaard splitting.

Examples:

(1) $S^3 = D^3 \cup D^3$

(2) $S^3 = S^1 \times D^2 \cup D^2 \times S^1$



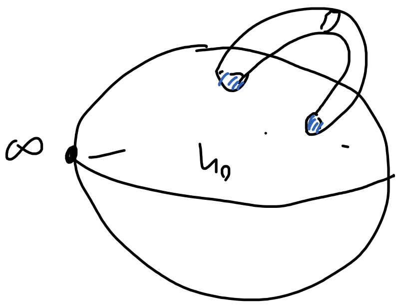
(3) $S^1 \times D^2 \cup S^1 \times D^2 = S^1 \times (D^2 \cup D^2) = S^1 \times S^2$

3.2. HEEGAARD DIAGRAMS

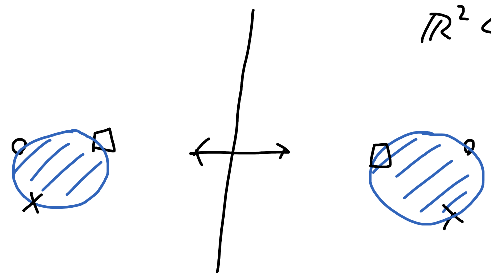
Consider: $\partial h_0 = \partial D^3 = S^2 = \mathbb{R}^2 \cup \{\infty\}$

Attaching region of 1-handle: $D^2 \sqcup D^2 \subset \mathbb{R}^2 \subset \partial h_0$

$\mathbb{R}^2 \subset \partial h_0$



$=$

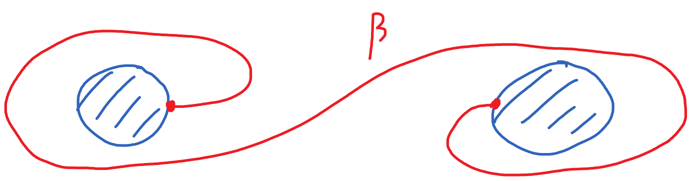


$(x,y) \longmapsto (-x,y)$

Attaching a 1-handle to ∂h_0 $\hat{=}$ gluing two disks $D^2 \sqcup D^2$ in $\mathbb{R}^2 \subset \partial h_0$ via an orientation reversing diffeo

* attaching sphere of 2-handle: $S^1 \subset \partial(h_0 \cup h_1^i)$

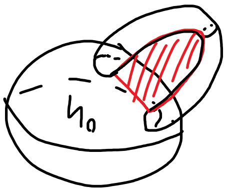
i.e. arcs $\beta_i \subset \mathbb{R}^2$ with endpoints on ∂D^2's

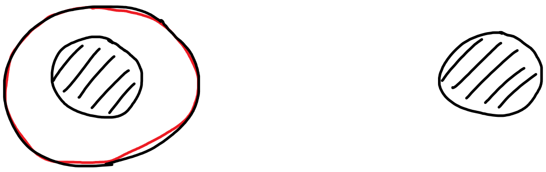


Def: $\mathbb{R}^2$ together with the attaching regions $D^2 \sqcup D^2$ of the 1-handles & attaching spheres β_i of 2-handles is called <u>(PLANAR) HEEGAARD DIAGRAM</u>

Ex: (1) $\emptyset$ (empty diag) $= h_0 \cup h_3 = S^3$

(2)  $\overset{\text{CANCEL}}{=} \emptyset = S^3$

$\left[\right.$  $\cup\, h_3 = S^3 \left.\right]$

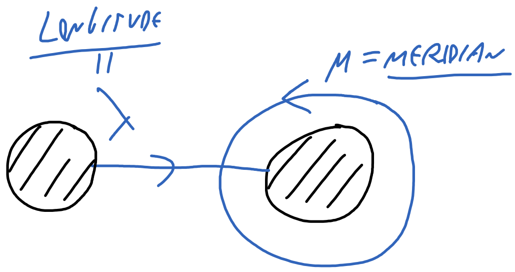
(3)  $= S^1 \times S^2$

$\left[\right.$ 

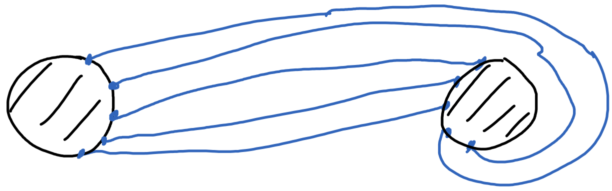
$S^1 \times D^2 \cup S^1 \times D^2 = S^1 \times S^2$ $\left.\right]$

(4) <u>lens spaces</u>: $L(p,q)$ $\quad p, q$ coprime

$L(p,q) :=$ genus-1-HS s.t. $\beta = -p\lambda + q\mu$



Ex: $L(5,2)$



THM 4:

A Heegaard diagram describes a unique handle decomp of unique 3-mfd M.

HEEGAARD GENUS:

$g(M^3) := \min \{ g(\Sigma) \mid \Sigma$ Heegaard surfaces of $M\}$

THM 6:

* $g(M^3) = 0 \iff M = S^3$
* $g(M) = 1 \iff M =$ LENS SPACES $\setminus S^3$
* $g(M_1 \# M_2) = g(M_1) + g(M_2)$ (HAKEN)

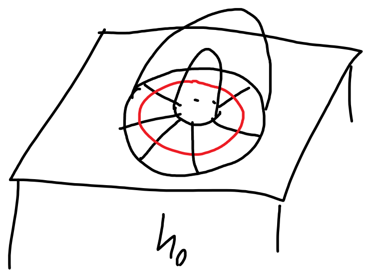
Thm 5:

A Heegaard diagram describes a unique handle decomp of a unique 3-mfd.

Proof:

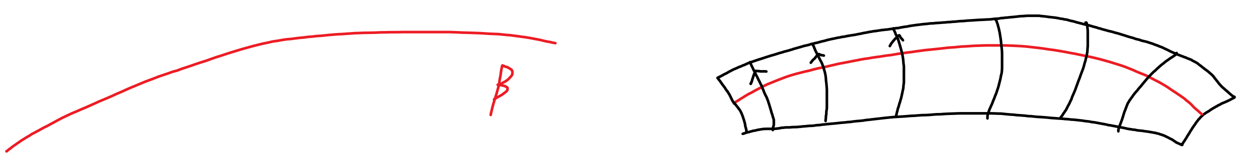
* $L 1 \Rightarrow$ Heeg. diag describes $M_1 = H_1$
* attaching map of a 2-handle:

$$\varphi : \partial D^2 \times D^1 \hookrightarrow \partial M_1$$



we know: $\varphi_0(\partial D^2 \times \{0\}) = \beta \subset \partial M_1$

$\subset \partial M_1$



$\{$framings of $\beta\} = \{ \beta = S^1 \longrightarrow GL_1(\mathbb{R}) = \mathbb{R}\setminus\{0\}\} = \mathbb{Z}_2$

$\Rightarrow$ Heegaard diag determines M_2

Lemma 6 (ALEXANDER TRICK)

$$\forall\, f : \partial D^n \xrightarrow[\cong]{C^0} \partial D^n \quad \exists\, F : D^n \xrightarrow[\cong]{C^0} D^n \text{ s.t. } F|_{\partial D^n} = f$$

* For $n = 1,2,3$: this is also true for C^∞ (MOISE) (SMALE)

Proof:

$$F : D^n \longrightarrow D^n$$
$$t \cdot x \longmapsto t \cdot f(x)$$
$$x \in \partial D^n \quad t \in [0,1]$$

∎

* attaching map of a 3-handle: $\varphi : \underset{\substack{\cong \\ S^2}}{\partial D^3} \times \{0\} \hookrightarrow \partial M_2$

M closed $\Rightarrow \partial M_2 = S^2$

$$\begin{tikzcd}
M \arrow[d, dotted, "\cong"'] & = & M_2 \arrow[d, "\cong"', "id"] & \underset{\varphi_1}{\overset{\cup_{\varphi_1}}{\longleftarrow}} \arrow[d, phantom, "\circlearrowright"] & h_3 \arrow[d, bend left] \\
M' & = & M_2 & \underset{\cup_{\varphi_2}}{\overset{\varphi_2}{\longleftarrow}} & h_3
\end{tikzcd}$$

extension of $\varphi_2^{-1} \circ \varphi_1$ (L.6)

$\rightarrow h_3 \rightarrow h_3$

∎

3.3. HANDLE SLIDES & STABILIZATIONS

$$\{\text{Heegaard diagrams}\} \longrightarrow\!\!\!\!\rightarrow \{3\text{-mflds}\}$$

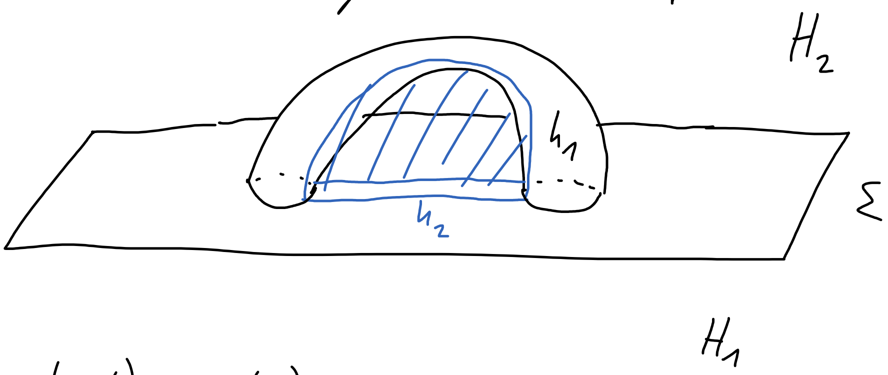
HANDLE CANCELLATION

$$M = (h_0 \cup h_1^1 \cup \dots \cup h_1^g) \cup_\Sigma (h_2^1 \cup \dots \cup h_2^g \cup h_3)$$

$$= (h_0 \cup h_1^1 \cup \dots \cup h_1^g \cup h_1^{g+1}) \cup_{\Sigma'} (h_2^1 \cup \dots \cup h_2^g \cup h_2^{g+1} \cup h_3)$$

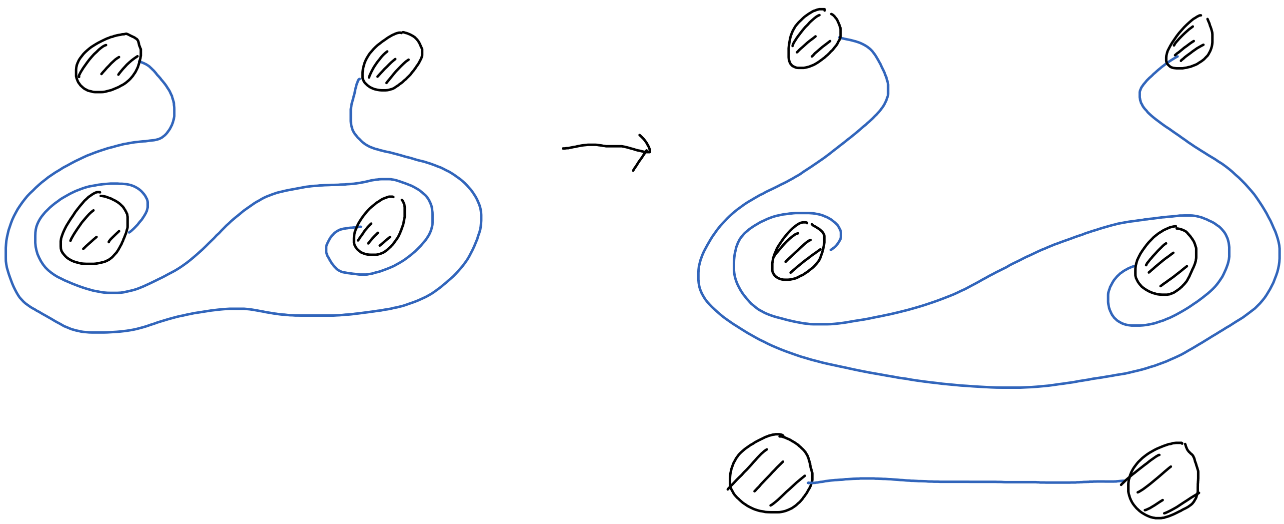
s.t. h_1^{g+1} & h_2^{g+1} cancel each other

STABILIZATION: (=) introducing a cancelling 1/2-handle pair

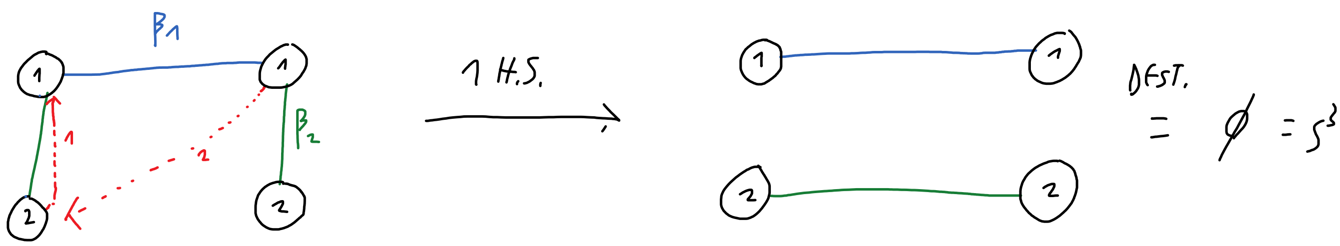


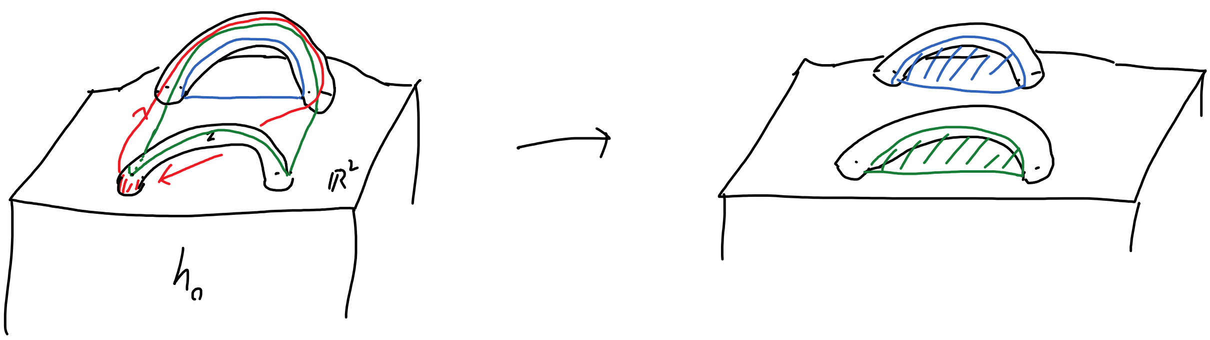
$$g(\Sigma') = g(\Sigma) + 1$$

In a Heegaard diagram:

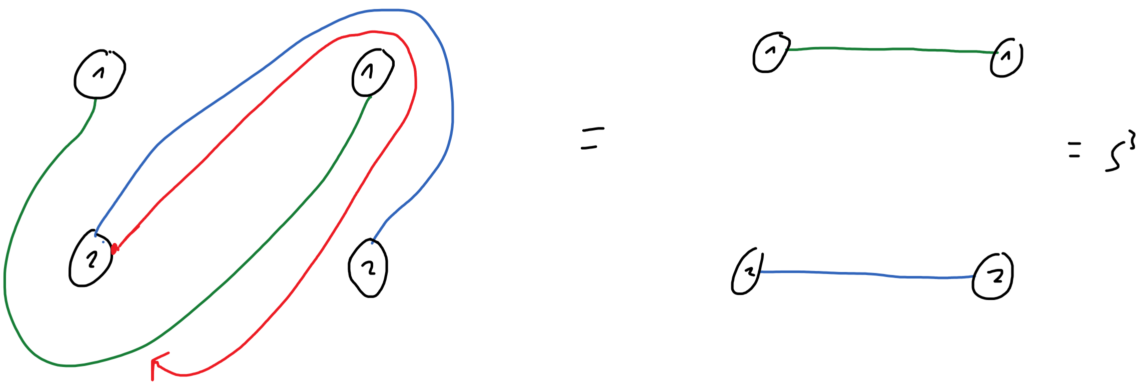


HANDLE SLIDES: (1-HANDLE SLIDES)



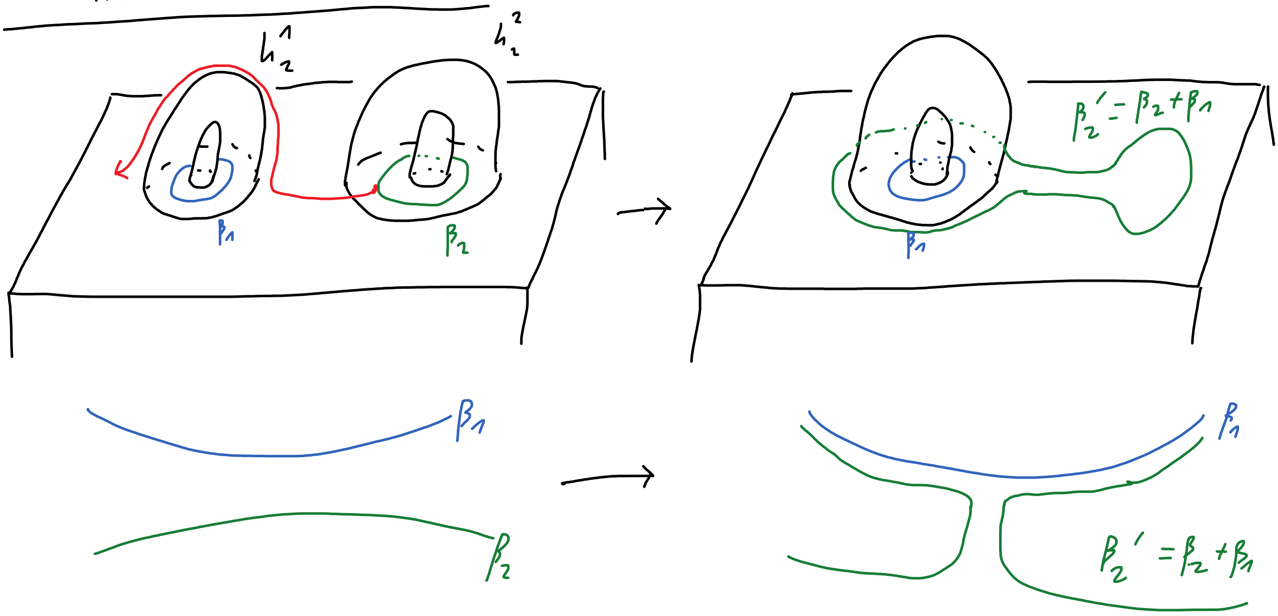


Ex: ISOTOPY OF 1-HANDLES

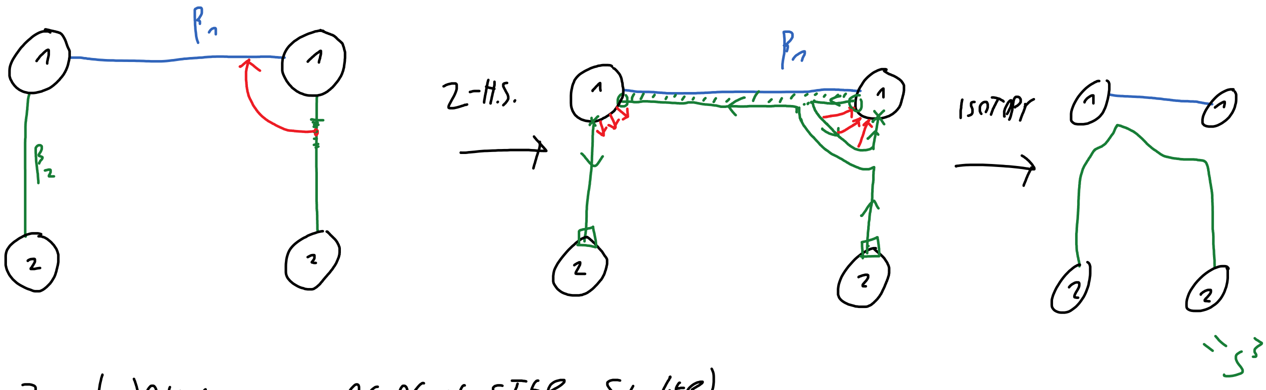


NOT a 1-handle slide

2-HANDLE SLIDES:



Ex:



THM 7 (JOHANNSON, REIDEMEISTER, SINGER)

$$\{\text{Heegaard diagrams}\} \Big/ \left\{\begin{array}{l}\text{stabilisations} \\ \text{1- \& 2-handle slides}\end{array}\right\} \xrightarrow{1:1} \{\text{3-mfds}\}$$

Proof: follows from T.2.6. (2) (CERF) $\square$

4. DIM 4: KIRBY DIAGRAMS:

4.1. KIRBY DIAGRAMS:

Let W^4 be a smooth, closed, orientable, connected 4-mfd with handle dec.

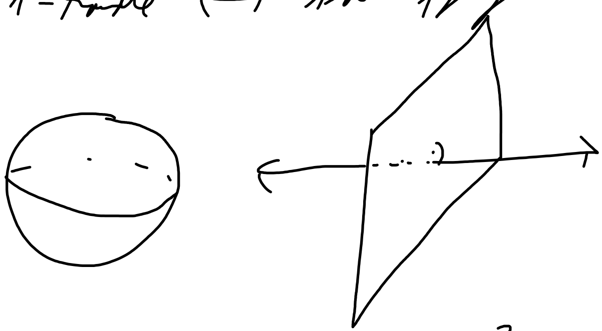
$$W^4 = \underbrace{\underbrace{h_0 \cup h_1^1 \cup \dots \cup h_1^{k_1}}_{W_1} \cup h_2^1 \cup \dots \cup h_2^{k_2}}_{W_2} \cup h_3^1 \cup \dots \cup h_3^{k_3} \cup h_4$$

* L.3.1. $\Rightarrow$ $W_1 \cong \natural_{k_1} S^1 \times D^3$

* Observe: $\partial h_0 = \partial D^4 = S^3 = \mathbb{R}^3 \cup \{\infty\}$

* <u>Attaching region of 1-handle</u>: $\partial D^1 \times D^3 = D^3 \sqcup D^3$

attaching a 1-handle $\longleftrightarrow$ identifying two D^3 via a reflection

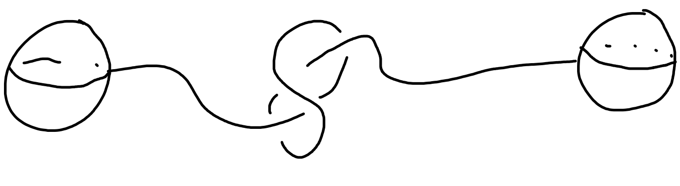


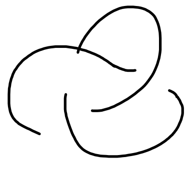
* <u>Attaching a 2-handle</u>: $h_2 = D^2 \times D^2$

$$\varphi : \partial D^2 \times D^2 \hookrightarrow \partial W_1$$

The attaching sphere $K := \varphi(\partial D^2 \times \{0\}) \subset \partial W_1$

is a knot in ∂W_1

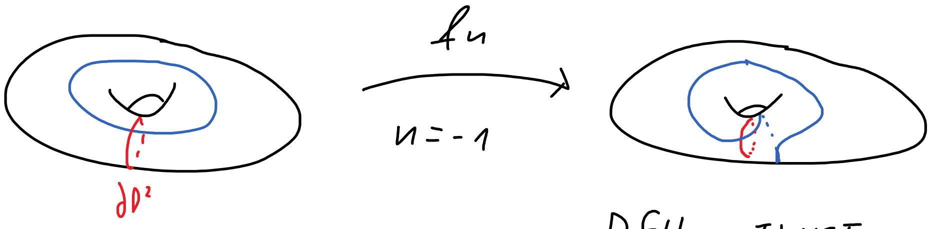




<u>Remark</u>: φ is <u>NOT</u> determined by K

$\ulcorner$ $S^1 \times D^2 \subset \mathbb{C}^2$ for $n \in \mathbb{Z}$: $\urcorner$

$$f_n : (e^{i\varphi}, re^{i\theta}) \longmapsto (e^{i\varphi}, re^{i(\theta + n\varphi)})$$



<u>DEHN-TWIST</u>

$\llcorner$ with $f_n(S^1 \times \{0\}) = S^1 \times \{0\}$ $\lrcorner$

<u>Def</u>: Let $K \subset M^3$ be an oriented knot in an oriented 3-mfd M^3.

A <u>FRAMING</u> of K is the choice of a diffeo

$$\varphi : S^1 \times D^2 \xrightarrow[]{\overset{C^\infty}{\cong}} \nu K \subset M \quad \text{s.t.}$$

$$S^1 \times \{0\} \longmapsto K$$

<u>Lemma 1</u>: $\{\text{framings of } K\}/_{\text{isotopy fixing } K} \xleftrightarrow{1:1} \pi_1(GL_2(\mathbb{R})) = \pi_1(O(2)) = \pi_1(S^1) = \mathbb{Z}$

<u>Proof</u>: We fix a framing $\varphi_0 : S^1 \times D^2 \xrightarrow{\overset{C^\infty}{\cong}} \nu K$

Let φ be another framing $\varphi : S^1 \times D^2 \xrightarrow{\overset{C^\infty}{\cong}} \nu K$

At a point $p \in S^1$ we have

$$\left(\varphi^{-1} \circ \varphi_0|_{p \times D^2} : D^2 \longrightarrow D^2\right) \in GL_2(\mathbb{R})$$

i.e. $\varphi \xmapsto{1:1} \{S^1 \longrightarrow GL_2(\mathbb{R})\}/_{\text{isotopy}}$ $\square$

Remark:

* This bijection is NOT canonical, i.e. it depends on φ_0
* If $\varphi_0 \hat{=} 0 \in \mathbb{Z}$ is given, then we get all other framings with

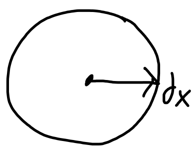
$$(f_n \circ \varphi_0) \hat{=} n \in \mathbb{Z}$$

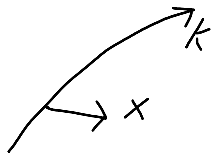
Lemma 2: $\{\text{framings of } K\} \xleftrightarrow{1:1} \{\text{VF } X \neq 0 \text{ def on } K, \text{ transverse to } K\}$

$$\xleftrightarrow{1:1} \{\text{knots } K' \text{ parallel to } K\}$$

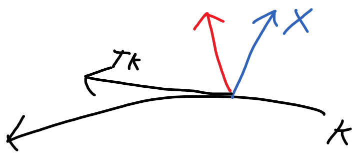
Proof: * Let $\varphi : \underset{(\theta,(x,y))}{S^1 \times D^2} \xrightarrow{\cong} \nu K$ be a framing

$X := T\varphi(\partial_x)$





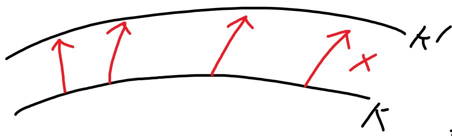
* Let $X \neq 0$ transverse to K



we get a basis of VF $\{TK, X, Y\}$

This induces a $\varphi : S^1 \times D^2 \xrightarrow{\cong} \nu K$

* $K' :=$ push-off of K in direction of X
* $X :=$ pointing in the direction of K'

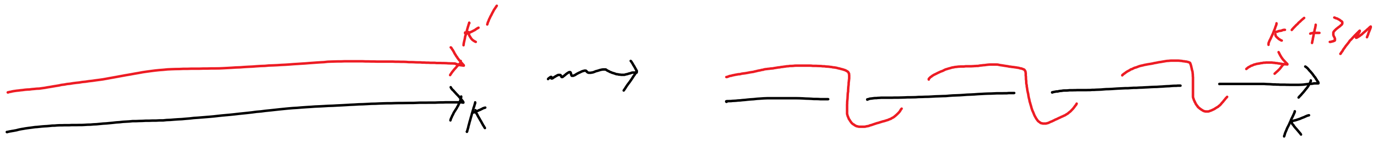


$\square$

i.e. we can describe framings of K by parallel knots K' &

if $K' \triangleq 0$

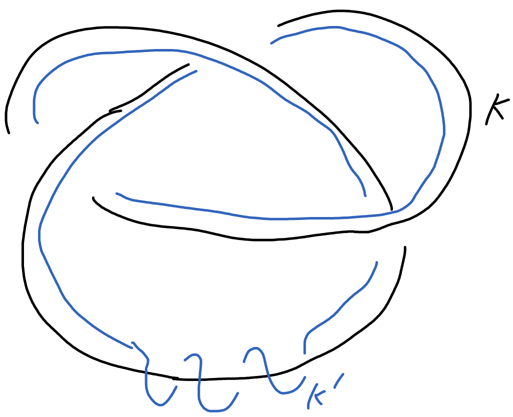
$\Rightarrow K' + n\mu \triangleq n$



where μ: 

is the MERIDIAN μ.

Ex:



$K \leftarrow$ attaching cycle of h_2 attached to ∂h_0

$\Rightarrow$ determines $h_0 \cup h_2$

Lemma 3: W is determined by W_2

Proof: $* \left(h_3^1 \cup \dots \cup h_3^{k_3}\right) \cup h_4 \overset{\text{dual h.d. \& L.3.1.}}{\cong} \natural_{k_3} S^1 \times D^3$

$* \; W^4$ closed $\Rightarrow \partial W_2 = \partial\left(\natural_{k_3} S^1 \times D^3\right) = \#_{k_3} S^1 \times S^2$

Thm 4 (LAUDENBACH - POENARU)

$\forall f: \#_k S^1 \times S^2 \xrightarrow{\overset{C^\infty}{\cong}} \#_k S^1 \times S^2$

$\Rightarrow \exists F: \natural_k S^1 \times D^3 \xrightarrow{\overset{C^\infty}{\cong}} \natural_k S^1 \times D^3$ s.t. $F|_\partial = f$

∎

THM 5

$\forall$ closed, oriented, smooth, connected 4-mfd W

$\exists$ Kirby diagr. uniquely describing W

Remark: A Kirby diagram also describes W_2 & ∂W_2.

Examples:

(1) $\emptyset$ (empty diag)

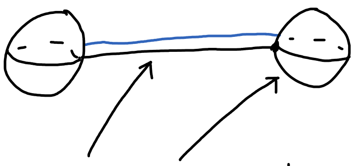
$W_2 = D^4$
$W = S^4$
$\partial W_2 = S^3$

(2) 

$W_2 = S^1 \times D^3$
$\partial W_2 = S^1 \times S^2$
$W = S^1 \times S^3$

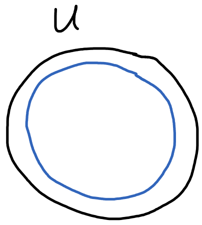
(3) 

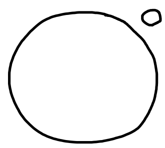
$W_2 = \natural_k S^1 \times D^3$
$\partial W_2 = \#_k S^1 \times S^2$
$W =$ later

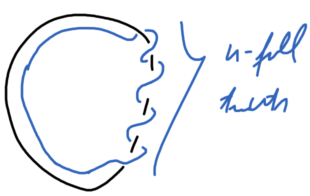
(4)  $= \emptyset$

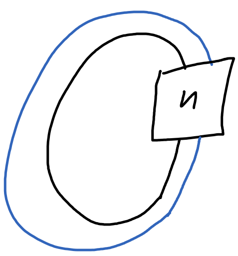
[$A_2 \pitchfork B_1 = \{pt\} \Rightarrow$ handles cancel]

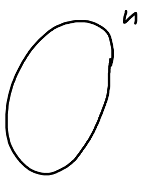
$W = S^4$
$W_2 = D^4$
$\partial W_2 = S^3$

(5) 

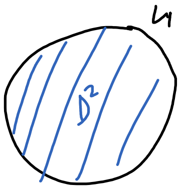
$=:$ 



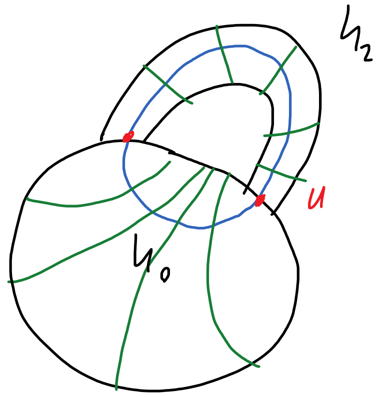
$=:$ 

$=$ 

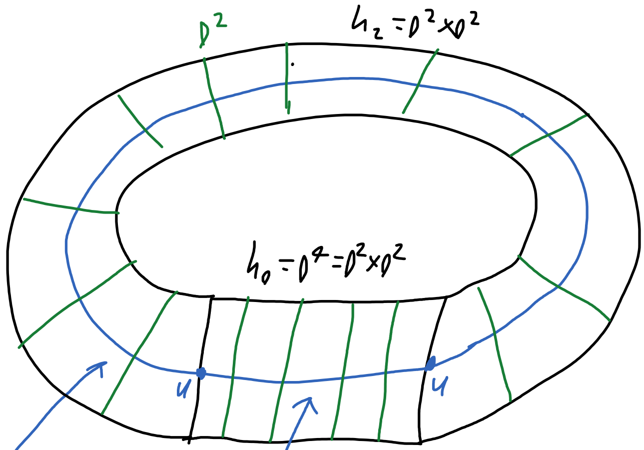
, $n \in \mathbb{Z}$



$= h_0 \cup h_2 = D^4 \cup D^2 \times D^2 = D^2 \times D^2 \cup D^2 \times D^2$



$=$



$S^2 = \underbrace{\text{core of } h_2}_{0 \times D^2} \cup \underbrace{\text{core disk of u pushed into } h_0}_{D^2}$

$\Rightarrow$ " $W_2 \overset{\text{locally}}{=} D^2 \times S^2$ "

$\Rightarrow W_2 = D^2$-bundle over S^2

FIBER BUNDLES:

$p: E \longrightarrow B$ surj s.t.

$\forall\, b \in B \;\exists$ open nbhd U s.t.

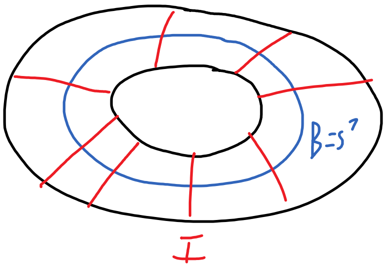
$$\begin{tikzcd} E \supset p^{-1}(U) \arrow[r, "\overset{C^\infty}{\cong}"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & U \times F \arrow[dl, "pr"] \\ B \supset U & \end{tikzcd}$$

E = TOTAL SPACE (here W_2)

B = BASIS (here S^2)

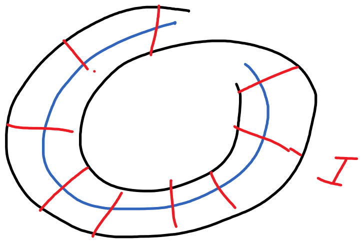
F = FIBER (here D^2)

Ex. (1)



$S^1 \times I = h_0 \cup h_1$

(2)



non-trivial $= h_0 \cup h_1$

(3) $M \times N$

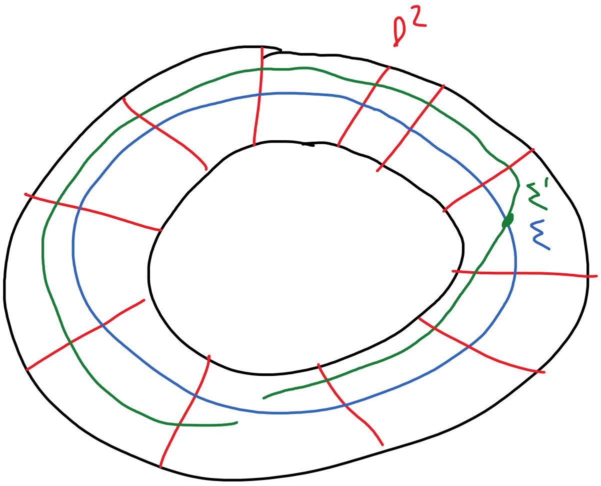
(4) $TM = \mathbb{R}^4$-bundle over M^4

* $TS^1 = \mathbb{R} \times S^1$
* TS^2 = non-trivial

(5) $DT\Sigma^2 = D^2$-bundle over Σ^2

(6) $\partial(D^2\text{-bundle over } \Sigma^2) = S^1$-bundle over Σ

EULER NUMBER: of an oriented D^2-bundle over Σ^2

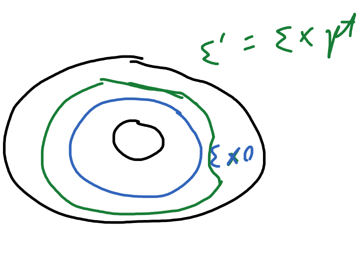


$e := \Sigma \cdot \Sigma' :=$

intersection number of Σ & Σ' with signs

where Σ' is isotopic to Σ & $\Sigma' \pitchfork \Sigma$

Ex: (1) $e(D^2 \times \Sigma) = 0$



$\Sigma' \pitchfork \Sigma = \emptyset \Rightarrow e = 0$

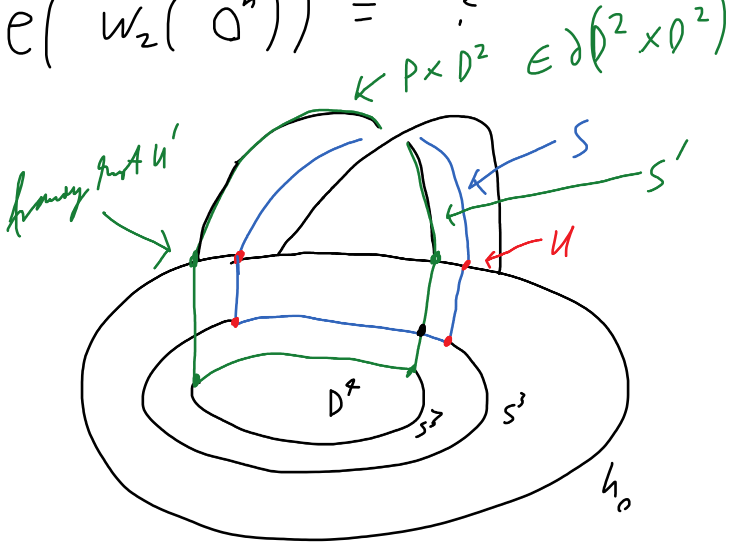
(2) $e(DT\Sigma_g) = 2-2g$

(POINCARÉ-HOPF INDEX THM)

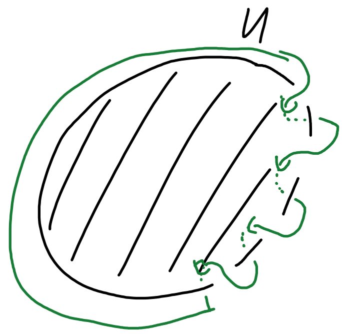
THM 6:

$\{\text{or. } S^1\text{-bundles over } \Sigma^2\} \xleftrightarrow{1:1} \{\text{or. } D^2\text{-bundles over } \Sigma^2\} \xleftrightarrow{1:1} e \in \mathbb{Z}$

$e(W_2(O^n)) = ?$



$\Rightarrow e(O^n) = S \cdot S' = S \cdot u' =$ Seifert surf. of $u \cdot u' = n$



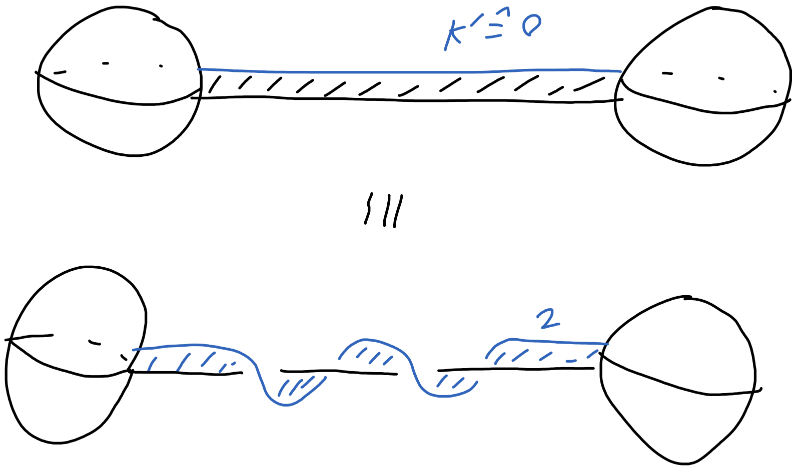
$\Rightarrow$ $\boxed{O^n = D^2\text{-bundle over } S^2 \text{ with } e = n}$

EXERCISE: $\partial(O^n) = L(n,1)$

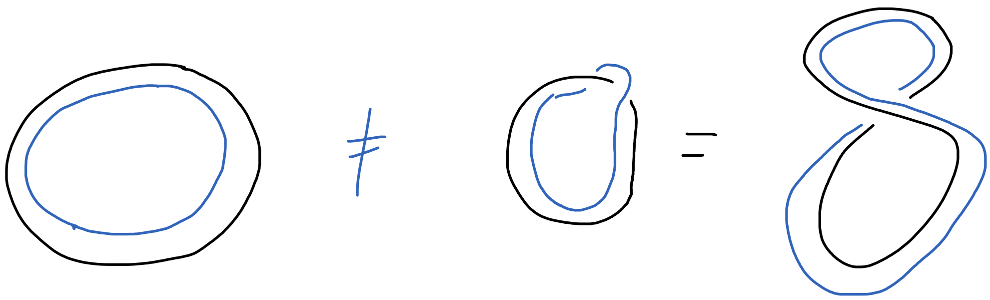
$\Rightarrow$ Only $O^{\pm 1}$, O^0 describe closed 4-mfds.

4.2. LINKING NUMBERS & FRAMINGS

Ex: (1)



(2)



BLACKBOARD FRAMING (not isotopy invariant)

GOAL: Find a isotopy invariant reference framing.

HERE: Consider handle decompositions of W^4 WITHOUT 1-handles

CONJ: $\pi_1(W^4) = 1 \Rightarrow \exists$ handle decomp. without 1-handles

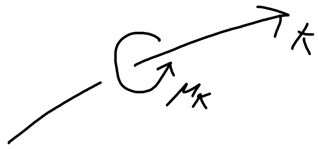
$\Rightarrow$ $\boxed{\text{Kirby diagrams of } W = \text{framed links in } S^3}$

Let $K \subset M$ an oriented knot in an oriented 3-mfd M.

Let K be nullhomologous, i.e.

$$[K] = 0 \in H_1(M) = H_1(M;\mathbb{Z})$$

$$\Rightarrow H_1(M \setminus \mathring{V}_K) \cong \mathbb{Z}_{\langle \mu_K \rangle} \oplus H_1(M)$$



* Let $K_1, K_2 \subset M$ be oriented & nullhomologous knots.

The LINKING NUMBER $lk(K_1,K_2) \in \mathbb{Z}$ is def by

$$[K_2] = lk(K_1,K_2) \cdot [\mu_{K_1}] \in H_1(M \setminus \mathring{V}_{K_1}) = \mathbb{Z}_{\langle \mu_{K_1} \rangle} \oplus H_1(M)$$

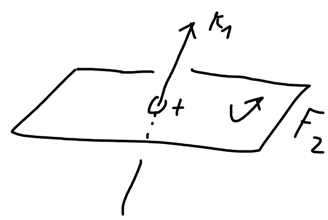
Remark: * $lk(K_1,K_2)$ is isotopy invariant.

* $lk(K_1,-K_2) = lk(-K_1,K_2) = -lk(K_1,K_2)$

Lemma 7:

(1) $K \subset M^3$ nullhomologous $\Leftrightarrow$ K admits a SEIFERT SURFACE F_K, i.e. F_K compact oriented surface $\subset M$ s.t. $\partial F_K = K$

(2) $lk(K_1,K_2) = K_1 \bullet F_2$, where F_2 is an arb. Seifert surface of K_2



Proof: (1) „$\Leftarrow$" $K = \partial F_K \Rightarrow [K] = 0$

„$\Rightarrow$" in S^3: (a) SEIFERT ALGORITHM

(b) $H_2(S^3 \setminus \mathring{V}K, \partial VK) \overset{PD}{=} H^1(S^3 \setminus \mathring{V}K) \overset{UKT}{=} F_1(S^3 \setminus \mathring{V}K) = \mathbb{Z}$

|| Homotopy

$[S^3 \setminus \mathring{V}K,\ K(\mathbb{Z},1) = S^1] \ni f_1$

$\boxed{F_K := f_1^{-1}(\text{reg value})}$

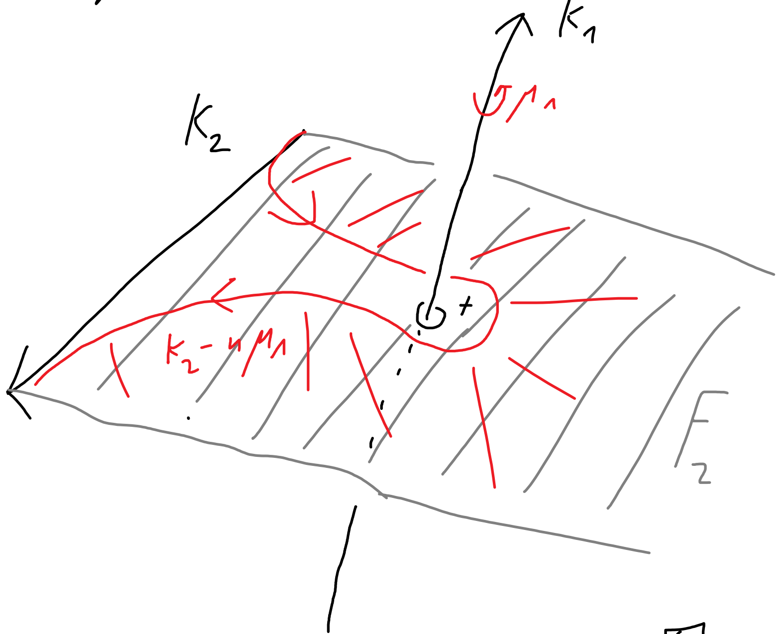
for general M: exercise

(2) Let $n := K_1 \cdot F_2$ (w.l.o.g. $n > 0$)

$\Rightarrow K_2 - n\mu_1$ bounds a homology class that does NOT intersect K_1

$\Rightarrow [K_2 - n\mu_1] = 0 \in H_1(M \setminus \mathring{V}K_1)$

$\Rightarrow lk(K_1, K_2) = n = K_1 \cdot F_2$



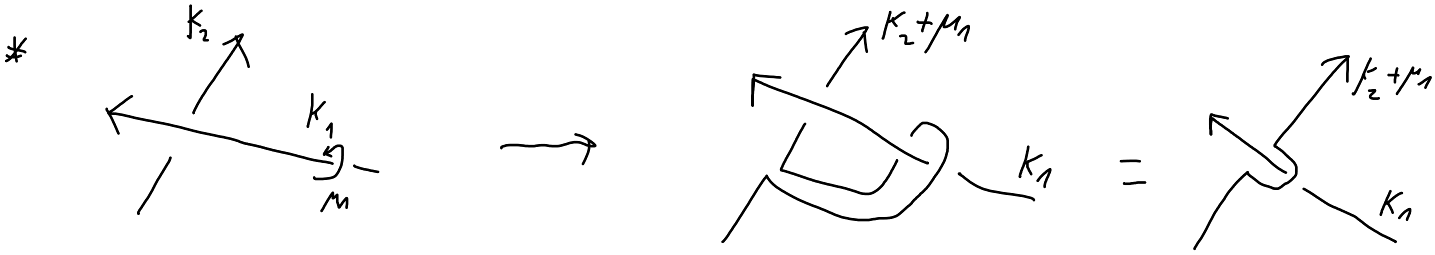
Lemma 8: Let $K_1, K_2 \subset S^3$

$\Rightarrow lk(K_1, K_2) = \#$ crossings of K_2 under K_1 with signs

Proof: $*$ $lk(K_1, \pm\mu_1) = \pm 1$

$*$ $lk(K_1, K_2 \pm \mu_1) = lk(K_1, K_2) + lk(K_1, \pm\mu_1) = lk(K_1, K_2) \pm 1$

$*$



Let $n :=$ # crossings of K_2 under K_1 with signs

$\Rightarrow K_2 - n\mu_1$ has NO undercrossings with K_1

$\Rightarrow lk(K_1, K_2 - n\mu_1) = 0$

$\Rightarrow lk(K_1, K_2) = n$ $\square$

Corollary 9: Let $K_1, K_2 \subset S^3$

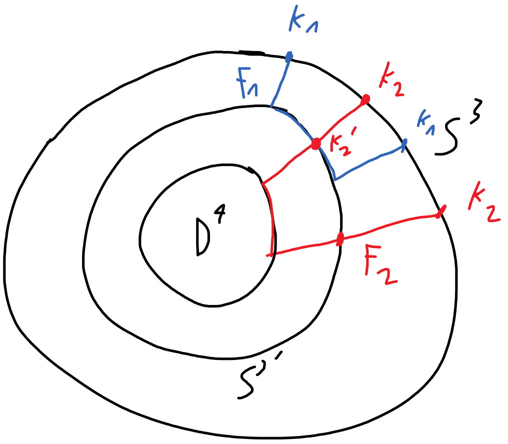
$\Rightarrow lk(K_1, K_2) = lk(K_2, K_1)$

Proof: Consider the diagram from "the other side" $\square$

Lemma 10 Let $K_1, K_2 \subset S^3 = \partial D^4$

$lk(K_1, K_2) = F_1 \cdot F_2$ with F_i Seifert surface of K_i in D^4.

Proof:

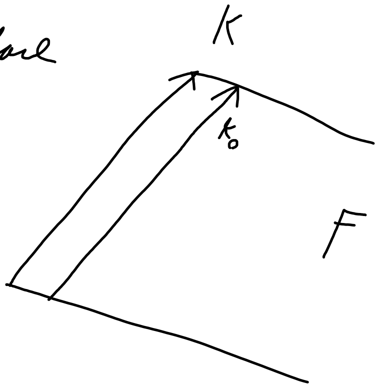


$$lk(K_1, K_2) = F_1 \cdot K_2 = F_1' \cdot K_2' = F_1 \cdot F_2 \qquad \blacksquare$$

Def: Let $K \subset M^3$ oriented, nullhomologous

* The parallel knot K_0 with $lk(K, K_0) = 0$ is called <u>SEIFERT / SURFACE FRAMING</u>, i.e.

  K_0 = push-off of K into a Seifert surface



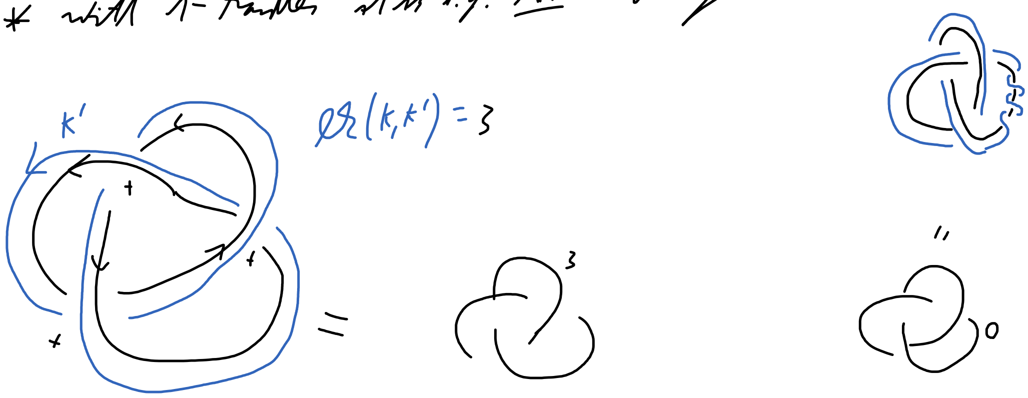
* Let K' be a framing of K

  $lk(K, K')$ is called <u>FRAMING COEFFICIENT</u>

Remark: * independent of isotopy

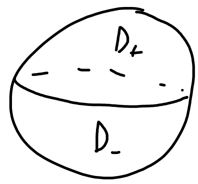
* with 1-handles it is i.g. NOT working

Ex:



Ex: (0) $\bigcirc^{+1} = \mathbb{C}P^2$

(1) $S^2 = \underset{\displaystyle h_0}{\underset{\shortparallel}{D_-}} \cup_\partial \underset{\displaystyle h_2}{\underset{\shortparallel}{D_+}}$



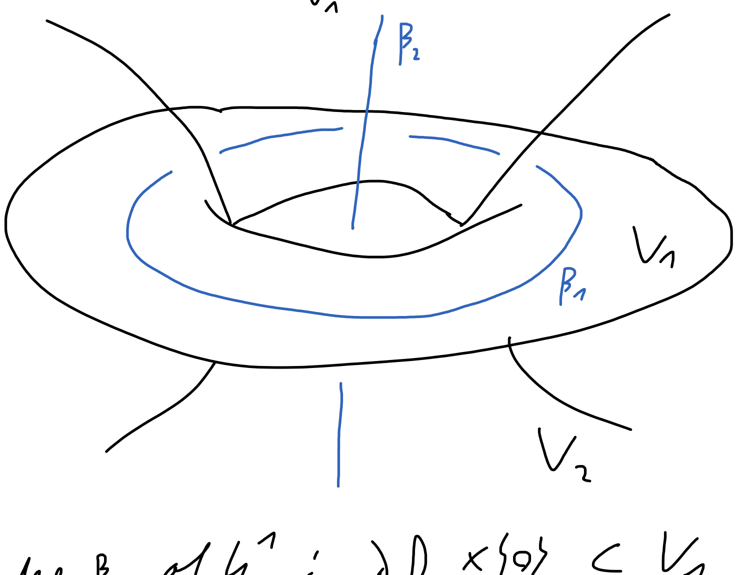
$$S^2\times S^2 = \underbrace{(D_-\times D_-)}_{h_0} \cup \underset{\displaystyle h_2^1}{\underset{\shortparallel}{(D_-\times D_+)}} \cup \underset{\displaystyle h_2^2}{\underset{\shortparallel}{(D_+\times D_-)}} \cup \underset{\displaystyle h_4}{\underset{\shortparallel}{(D_+\times D_+)}}$$

(i.e. $h_k^{(n)} \times h_\ell^{(m)} = h_{k+\ell}^{(n+m)}$)

we observe: $\underbrace{(D_-\times D_-)}_{h_0} \cup \underbrace{(D_-\times D_+)}_{h_2^1} = D_-\times S^2 = \bigcirc^0$

$\underbrace{(D_-\times D_-)}_{h_0} \cup \underbrace{(D_+\times D_-)}_{h_2^2} = S^2\times D_- = \bigcirc^0$

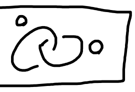
$S^3 = \partial h_0 = \partial(D_-\times D_-) = \underbrace{(\partial D_-\times D_-)}_{V_1} \cup \underbrace{(D_-\times \partial D_-)}_{V_2}$ = genus-1 Heegaard splitting of S^3



HOPF-LINK

attaching sphere β_1 of h_2^1: $\partial D_- \times \{0\} \subset V_1 \subset S^3 = \partial h_0$

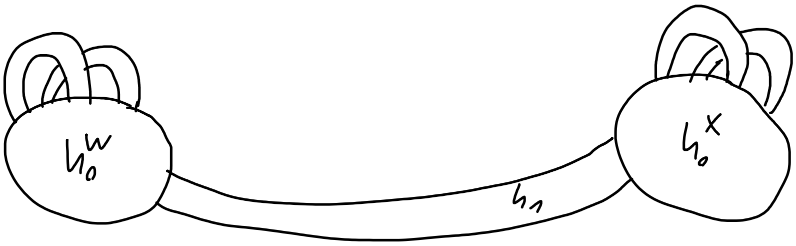
" β_2 " h_2^2: $\{0\}\times \partial D_- \subset V_2 \subset S^3 = \partial h_0$

$S^2\times S^2 =$ 

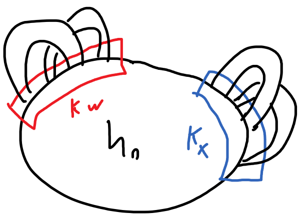
Let W, X be closed 4-mfds with Kirby diagrams K_W & K_X.

(2) $W_2 \natural X_2 = K_W \sqcup K_X := \boxed{K_W}\boxed{K_X}$

$\partial W_2 \# \partial X_2 = K_W \sqcup K_X$

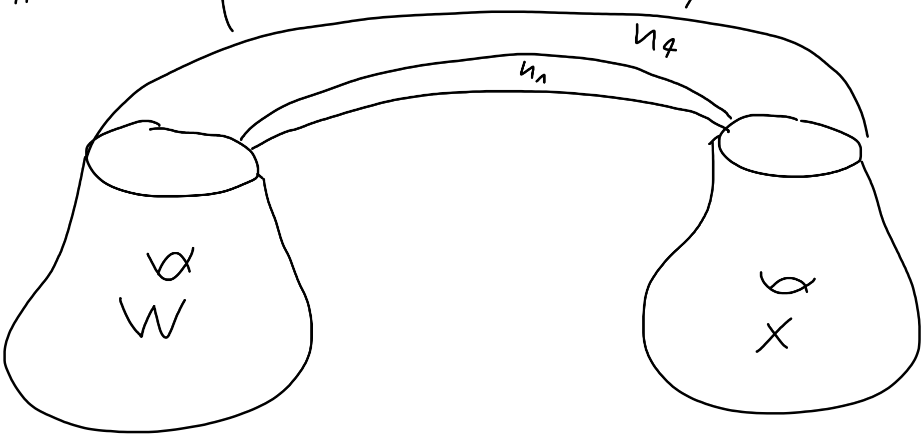


|| cancellation

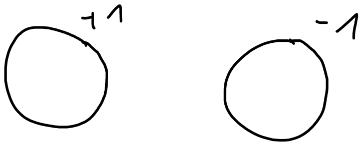


S^2 = belt sphere of 1-handle

(3) $W \# X = (W \setminus h_4 \natural X \setminus h_4) \cup h_4$



$= K_W \sqcup K_X$

(4) $W =$ 

$= \mathbb{C}P^2 \# -\mathbb{C}P^2 \overset{\text{(later)}}{\cong} S^2 \tilde{\times} S^2$

$W_2 = (D^2\text{-bundle over } S^2 \text{ with } e=+1) \natural$
$\qquad (\quad '' \qquad e=-1)$

$\partial W_2 = S^3 \# S^3 = S^3$

4.3. THE INTERSECTION FORM & HOMOLOGY OF A 2-HANDLEBODY

Let W^4 be a compact, oriented, smooth 4-mfd.

Lemma 11

$\forall a \in H_2(W)$ $\exists$ smooth or. surface $\Sigma_a^2 \subset W$ s.t. $a = [\Sigma_a^2]$

Proof: in the case of $W =$ 2-HANDLEBODY, i.e. $W = h_0 \cup \{h_2^i\}$

general case: exercise

(a) $\pi_1(W) = 1$

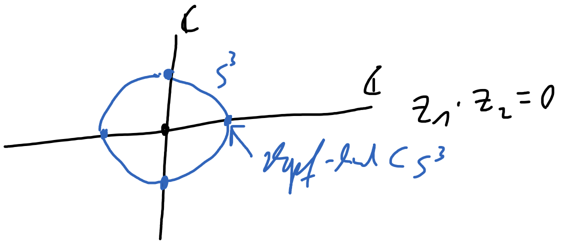
$\overset{\text{HUREWICZ}}{\Rightarrow}$ $H_2(W) \cong \pi_2(W)$

Let $a \in H_2(W)$

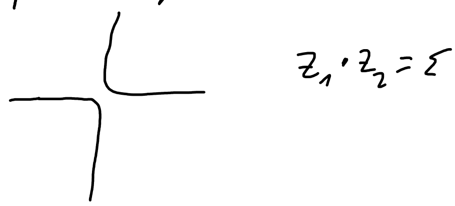
$\Rightarrow \exists$ immersion $f: S^2 \looparrowright W$ with finitely many double points $p_1, \dots, p_k$ s.t. $[f(S^2)] = a$

local model of double point:

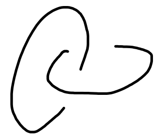
$$z_1 \cdot z_2 = 0 \quad \text{in } \mathbb{C}^2$$



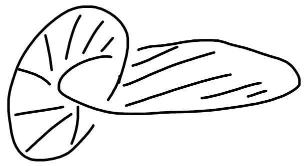
Idea:
replace by



Consider: $\{z_1 \cdot z_2 = 0\} \cap S^3 = $ Hopf link



Replace $\{z_1 \cdot z_2 = 0\} \cap D^4$ by



$\Rightarrow$ get embedded surface Σ_α of genus $= k$ ▨

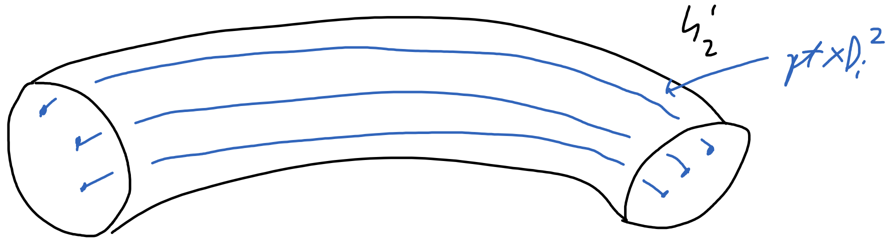
(b) HANDLE DECOMP:

$$H_2(W) = \langle h_2^1, \ldots, h_2^k \rangle_{\mathbb{Z}} \cong \mathbb{Z}^k$$

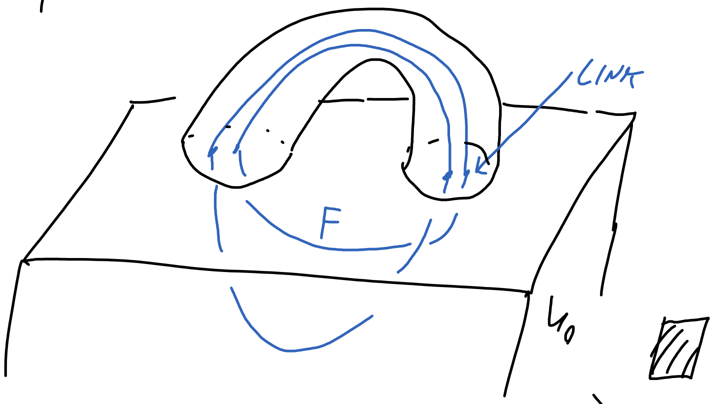
Let $\alpha \in H_2(W)$

$$\Rightarrow \quad \alpha = \sum_{i=1}^{k} c_i \; h_2^i$$

Start with c_i disjoint copies of $pt \times D_i^2 \subset h_2^i$



$\Rightarrow \partial\left(\sum_{i=1}^{k} c_i\,(pt \times D_i^2)\right) \subset S^3 = \partial h_0$ is a nullbordant link, i.e. bounds a Seifert surface F



$\Sigma_a = F \cup \sum_{i=1}^{k} c_i\,(pt \times D_i^2)$

$$\left(H_2(W,\partial W) \overset{PD}{=} H^2(W) = [W, K(\mathbb{Z},2)] \overset{?}{=} [W, \mathbb{C}P^2] \right)$$

$K(\mathbb{Z},2) \cong \mathbb{C}P^\infty$

Def: INTERSECTION FORM

$$Q_W : H_2(W) \times H_2(W) \longrightarrow \mathbb{Z}$$
$$(a,\ b) \longmapsto \Sigma_a \cdot \Sigma_b$$

Lemma 12:

(1) Q_W is well-def.

in part. $e(D^2\text{-bundle over } \Sigma) = Q_W$ is well-def

(2) $Q_W = 0$ on torsion

(3) $Q_W : H_2/_{Tor} \times H_2/_{Tor} \longrightarrow \mathbb{Z}$ is a sym. bilinear form

& represented by a symmetric matrix M

(4) * $\det(Q_W) = \det(M) = \pm 1 \iff$: Q_W is UNIMODULAR

$\iff$ ∂W is a homology sphere or $\partial W = \emptyset$

* $\partial W = \emptyset \Rightarrow \tau(Q_W) = \#$ pos eigenvalues of M − $\#$ neg " of M

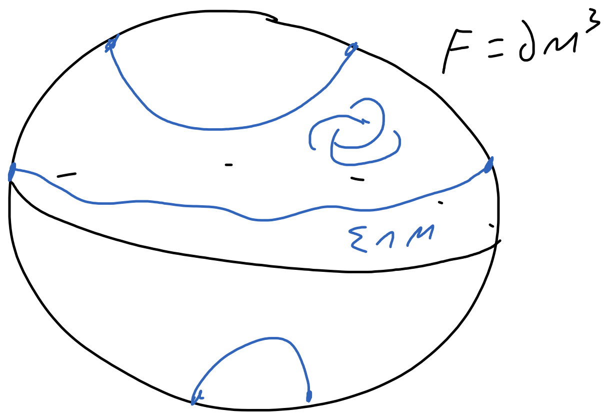
SIGNATURE $= b_2^+ - b_2^-$

(5) Q_W is also def on top mfds.

Proof: (1) Let $F^2 = \partial M^3 \subset W^4$ & $\Sigma^2 \subset W^4$

To show: $\Sigma \cdot F = 0$

w.l.o.g. $\Sigma \cap M \subset M$ is a 1-mfd



$\Rightarrow$ $\boxed{\Sigma \cdot F = 0}$

(2) & (3) lin. alg.

(4) Exercise

(5) $H_2(W) \cong H^2(W, \partial W)$

$H^2(W,\partial W) \times H^2(W,\partial W) \longrightarrow \mathbb{Z}$

$(\alpha, \beta) \longmapsto \alpha \cup \beta$

$\boxtimes$

Ex: (1) $Q_{S^4} = 0$ $\qquad (H_2 = 0)$

(2) $Q_{D^2\text{-bundle over } \Sigma_g} = (e)$ for the basis given by $H_2 = \langle \Sigma_g \rangle_{\mathbb{Z}}$

$Q_{\pm \mathbb{CP}^2} = (\pm 1)$

(3) $Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the basis $H_2 = \langle S^2 \times pt, pt \times S^2 \rangle$

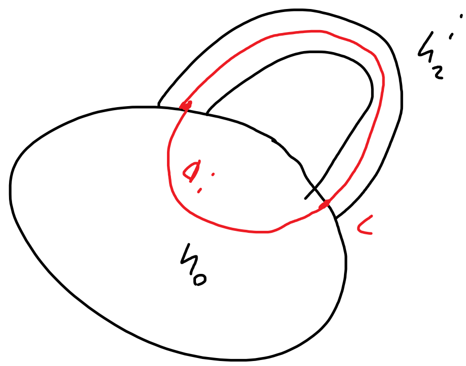
Lemma 14:

Let W^4 be a 2-handlebody, represented by a framed, oriented link $L = L_1 \cup \dots \cup L_n$ with framings $f_1, \dots, f_n$

$\Rightarrow$ * $H_2(W) \cong \mathbb{Z}^n$ with basis given by

$\alpha_i = [\, F_i \cup \text{core of } h_2^i \,]$ where F_i = Seifert surface of L_i

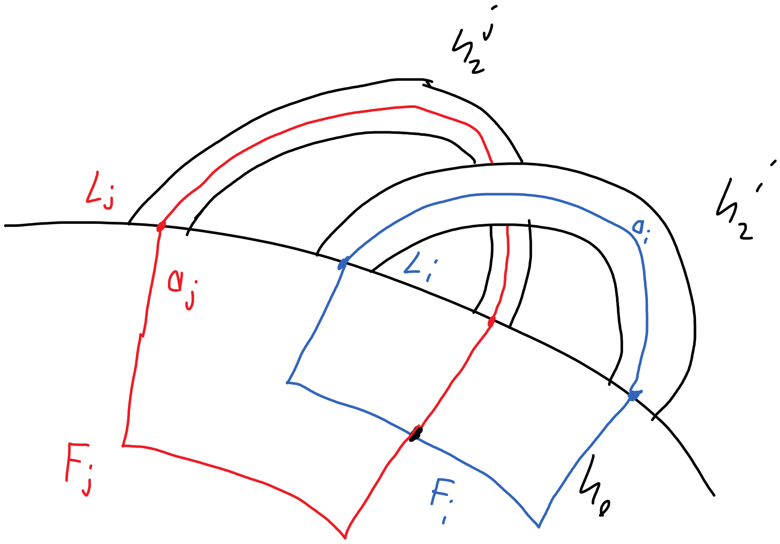
* Q_W is represented in the basis $\langle \alpha_1, \dots, \alpha_n \rangle$ by LINKING MATRIX:



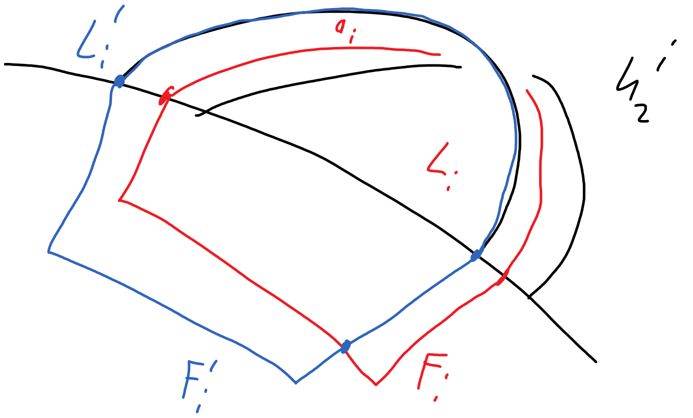
$$M := \begin{pmatrix} f_1 & & \mathrm{lk}(L_i, L_j) \\ & \ddots & \\ \mathrm{lk}(L_j, L_i) & & f_n \end{pmatrix}$$

Proof:

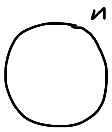
$i \neq j$



$$a_i \cdot a_j = F_i \cdot F_j = F_i \cdot L_j = lk(L_i, L_j)$$



$$a_i \cdot a_i = F_i' \cdot F_i = L_i' \cdot F_i = lk(L_i', L_i) = f_i$$ ∎

Ex: (1) 

$Q_W = (n)$

(2) 

$= S^2 \times S^2$ $\quad Q_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(3) 

$Q_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow \partial W_2 =$ homology sphere

4.4. TOPOLOGICAL 4-MFDS

Observe: $\pi_1(W^4)=1 \Rightarrow H_1 = H_3 = 0$ & $H_2 = \mathbb{Z}^n$

$\longrightarrow$ all inf. in Q_W

Def: Q_W is EVEN $:\Leftrightarrow$ $Q_W(\alpha,\alpha) \equiv 0 \mod (2) \quad \forall \alpha$

Q_W is ODD $:\Leftrightarrow$ else

Thm 15 (FREEDMAN)

$\forall$ Q unimodular, sym. bilinear form

$\exists$ TOPOLOGICAL closed 4-mfd W with $\pi_1(W)=1$ & $Q_W = Q$

* Q even $\Rightarrow$ W is unique
* Q odd $\Rightarrow$ $\exists$ exactly two such W, at least one does NOT carry a smooth str.

Corollary 16

$$X^4 \simeq S^4 \Leftrightarrow \pi_i(X^4) \cong \pi_i(S^4)$$

$$\Leftrightarrow \pi_1(X^4) = 1 \quad H_*(X^4) \cong H_*(S^4)$$

$$\Leftrightarrow X^4 \overset{C^0}{\cong} S^4 \qquad \blacksquare$$

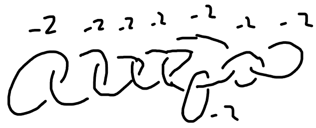
Thm 17 (WALL)

Let W^4, X^4 closed, smooth with $\pi_1 = 1$ & $W^4 \overset{C^0}{\cong} X^4$

$\Rightarrow \exists K \in \mathbb{N}_0 : W^4 \#_K S^2\times S^2 \overset{C^\infty}{\cong} X^4 \#_K S^2 \times S^2$

$\boxed{Q}$ Is $K=1$ enough?

A TOPOLOGICAL 4-MFD WITHOUT SMOOTH STR.

$P_{E_8} =$ 

a compact smooth 4-mfd (c.f. SHEET 5)

$$Q_{P_{E_8}} = E_8 = \begin{pmatrix} -2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & -2 & -1 \\ & & & -1 & \ddots \end{pmatrix}$$

$\det(E_8) = 1$

$\sigma(E_8) = 8$

E_8 is even

$\Rightarrow \partial P_{E_8}$ is a homology sphere, i.e. $H_*(\partial P_{E_8}) = H_*(S^3)$

THM 18 (FREEDMAN 80')

Let M^3 be a homology sphere

$\Rightarrow \exists$ a topological contractible 4-mfd Δ^4_M s.t. $\partial \Delta_M = M$

(FAKE 4-BALL)

$\hat{E}_8 := P_{E_8} \cup \Delta_{-\partial P_{E_8}}$ a closed top 4-mfd.

THM 19 (ROKHLIN 50')

Let W^4 be smooth closed 4-mfd with $w_2(TW) = 0$ (SPIN)

$\Rightarrow \sigma(W) \equiv 0 \mod (16)$

(see SCORPAN / KIRBY for a proof)

COR 20 $\hat{E}_8$ does NOT carry a smooth str.

Proof: $\sigma(\hat{E}_8) = 8 \not\equiv 0 \mod(16)$ & E_8 even $\Rightarrow w_2(\hat{E}_8) = 0$

⌈ $w_2 \in H^2(W;\mathbb{Z}_2) = H_2(W;\mathbb{Z}_2)$ represent by $\tilde{w}_2 \in H_2(W;\mathbb{Z})$ ⌉

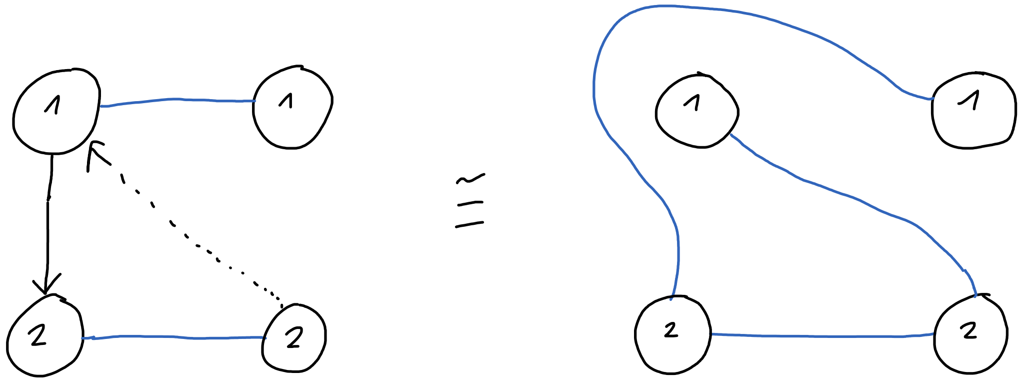
⌊ $\Rightarrow \tilde{w}_2 \cdot x \equiv x \cdot x \mod(2)\ \forall x \in H_2(W) \overset{Q \text{ even}}{\Rightarrow} \tilde{w}_2 \equiv 0 \mod(2) \Rightarrow w_2 = 0$ ⌋ ∎

5. KIRBY CALCULUS

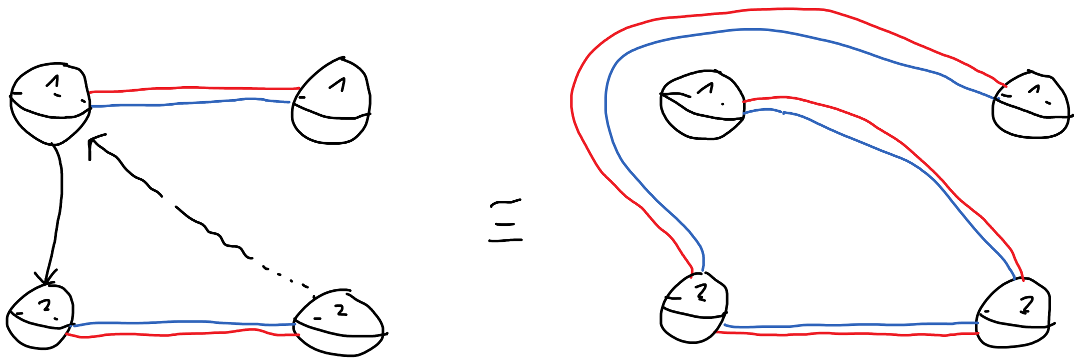
5.1. HANDLE SLIDES

1-HANDLES

DIM=3:

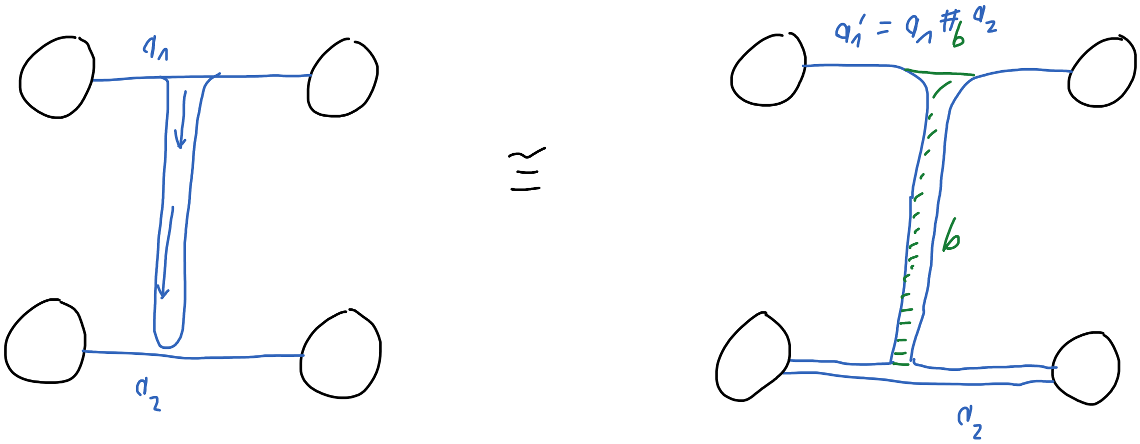


DIM=4:



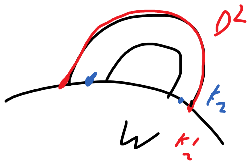
2-HANDLES:

DIM=3:



DIM = 4:

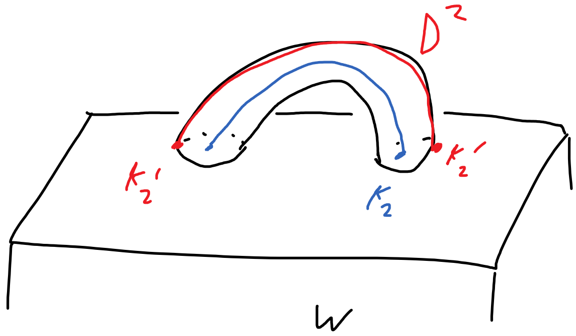
Let (K_1, K_1') & (K_2, K_2') be framed knots in $M^3 = \partial W$ along which we attach 2-handles h_2^1 & h_2^2



$\Rightarrow$ K_2' bounds a disk in $\partial(W \cup h_2^2)$

2-handle slide of h_2^1 over h_2^2:

Move (K_1, K_1') over D^2 with $\partial D^2 = K_2'$

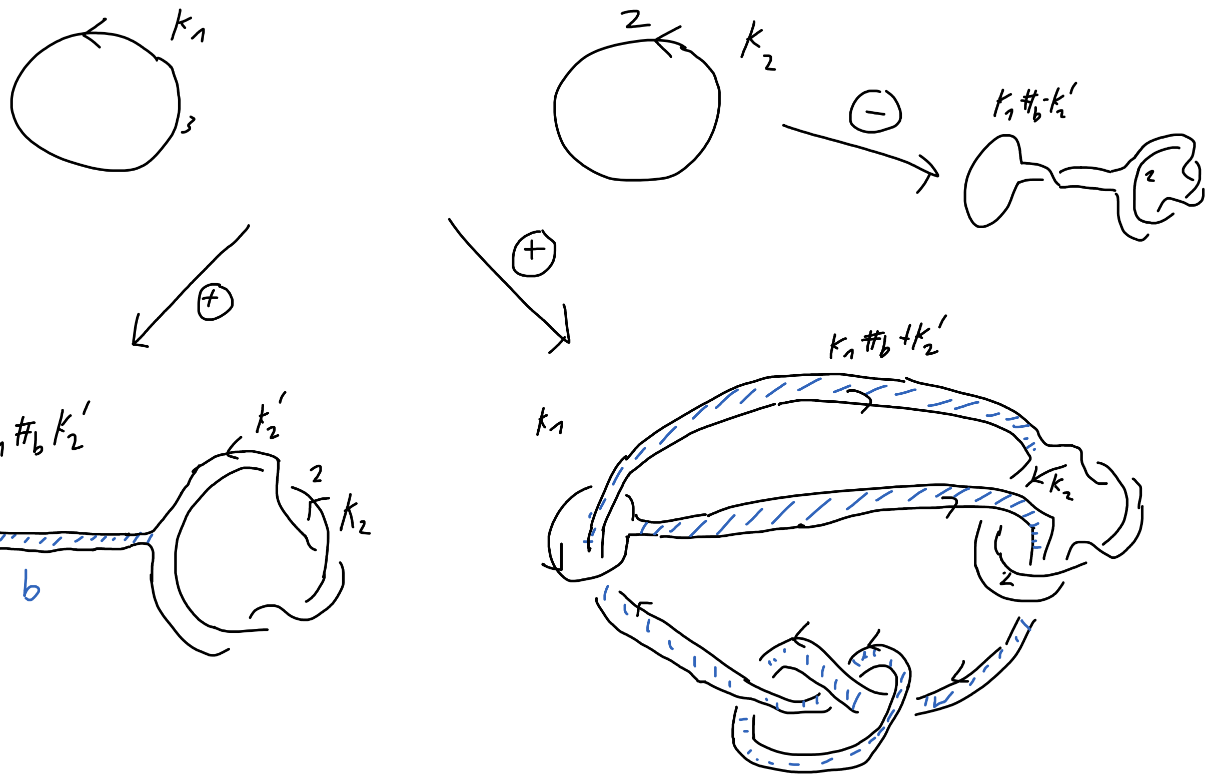


$\rightarrow$ take $K_1 \#_b (\pm K_2')$ along a band b

+ = 2-HANDLE ADDITION

− = 〃 SUBTRAKTION

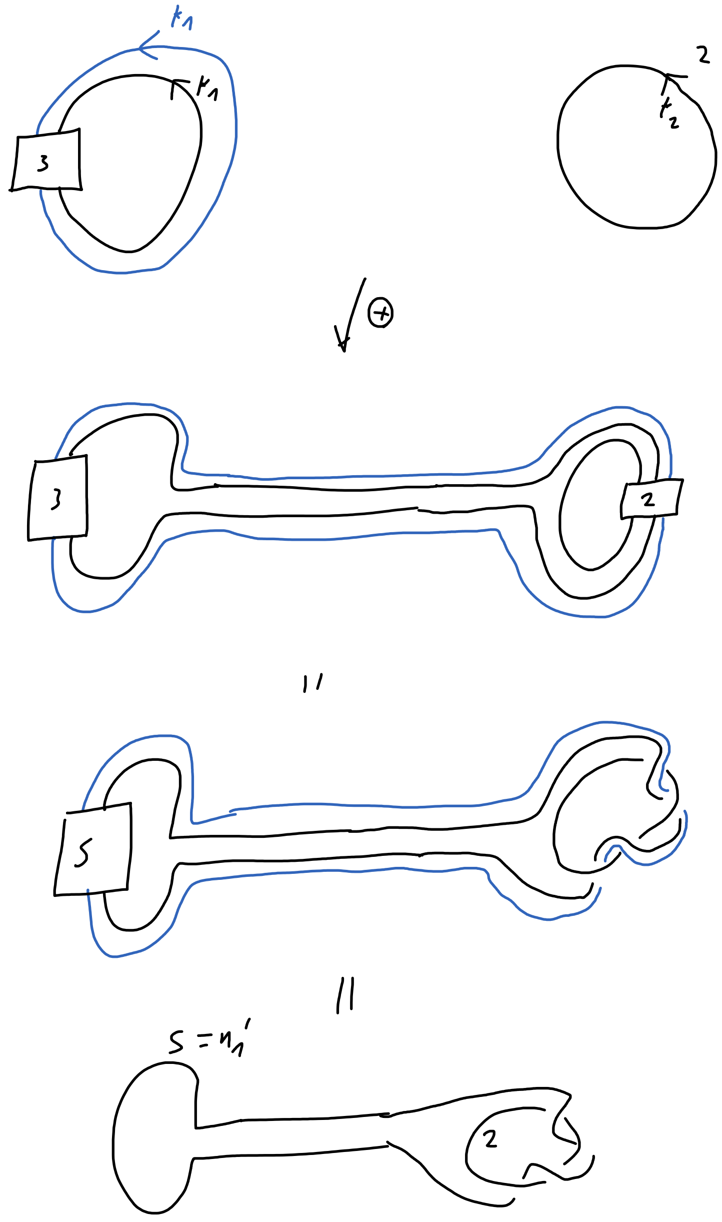
Ex:



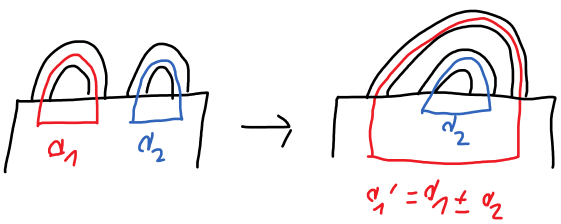
$\boxed{\text{WHAT HAPPENS TO THE FRAMING?}}$

FRAMINGS:

(1) Draw parallel knots!



(2) framing itself:



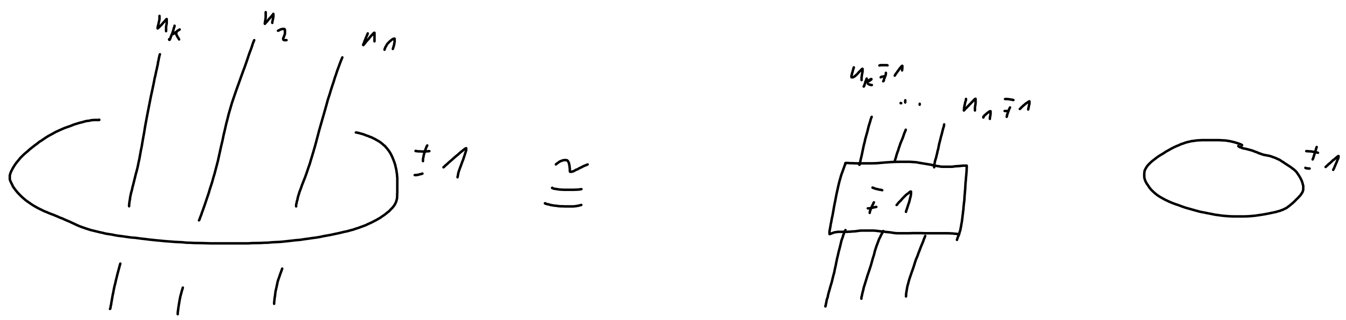
Let W be a 2-handle body

$a_1, \dots, a_n$ basis of $H_2(W)$ given by 2-handles

$$n_1' = (a_1 \pm a_2)\cdot(a_1 \pm a_2) = a_1\cdot a_1 \pm 2\, a_1\cdot a_2 + a_2\cdot a_2 = n_1 \pm 2\,\mathrm{lk}(k_1,k_2) + n_2$$

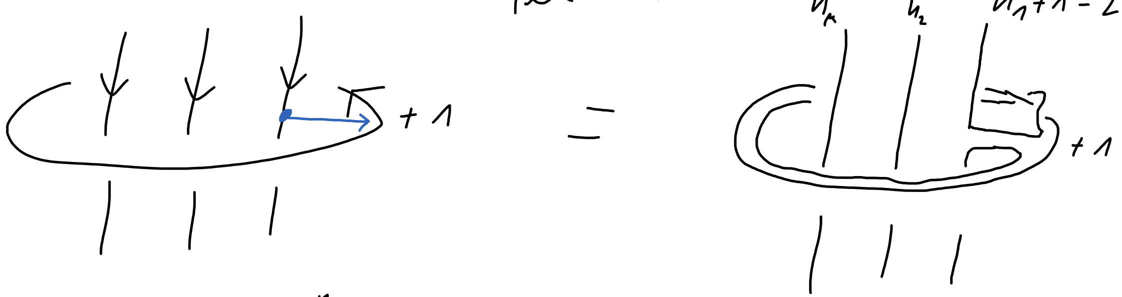
new framing: $\boxed{n_1' = n_1 + n_2 \pm 2\,\mathrm{lk}(k_1, k_2)}$

Lemma 1:

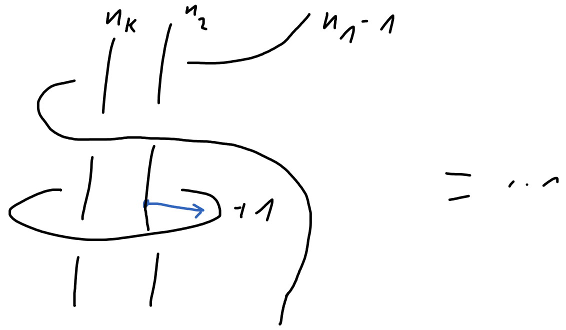


Proof:

($\varepsilon = -1$)



Isotopy
=



$= \ldots$

∎

Corollary 2

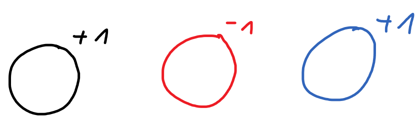
$(S^2 \times S^2) \# \mathbb{CP}^2 \cong \mathbb{CP}^2 \# (-\mathbb{CP}^2) \# \mathbb{CP}^2$

(c.f. $T^2 \# \mathbb{RP}^2 \cong \#_3 \mathbb{RP}^2$)

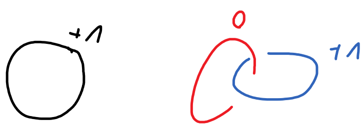
Proof:

$\mathbb{CP}^2 \# (-\mathbb{CP}^2) \# \mathbb{CP}^2$

=



L.1
=



L.1
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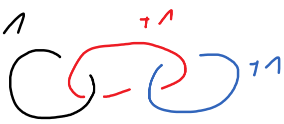

L.1
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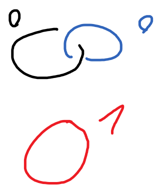


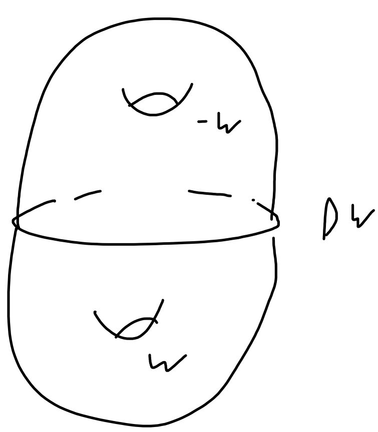
$= S^2 \times S^2 \# \mathbb{CP}^2$

∎

DOUBLES:

Let W^4 be compact oriented with $\partial W \neq \emptyset$ without 3- & 4-handles

$$DW := W \cup_{id_{\partial W}} (-W)$$



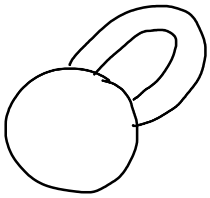
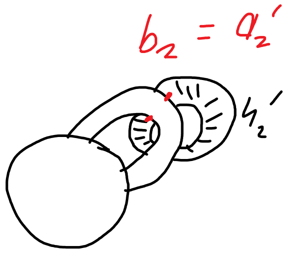
$DW = W \cup$ dual handles

k-handle of $W \longrightarrow (4-k)$-handle of $-W$

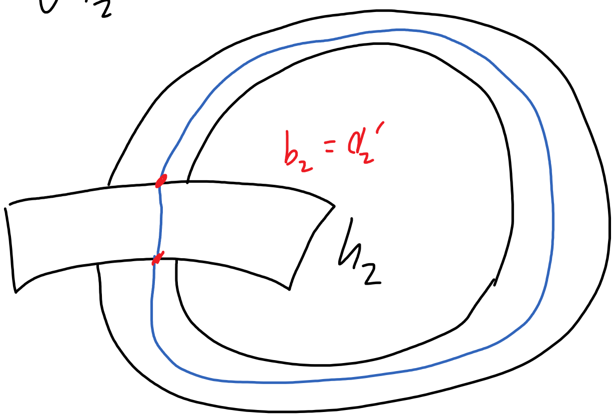
2-handle h_2 of $W \longrightarrow$ 2-handle h_2' of $-W$

h_2' is dual to h_2

$\Rightarrow h_2'$ is attached along the belt sphere b_2 of h_2

$W =$  $\longrightarrow DW$ 

$* \; h_2 \cup h_2' = D^2 \times S^2$



$S^2 =$ core of $h_2' \cup$ co-core of h_2

framing = product framing = 0

i.e. $W \longrightarrow DW \;\hat{=}\;$ add 0-framed meridians to the 2-handles

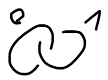
Ex: $W_2 =$ 

$\Rightarrow DW_2 =$ 

S^2-bundles over S^2

Thm 3: (up to diffeo/or homeo)*

(1) $\exists$ exactly two S^2-bundles over S^2:

$S^2 \times S^2$ =  $S^2 \tilde{\times} S^2$ =  (TWISTED BUNDLE)

(2) $D($ D^2-bundle over S^2 with even Euler number$)$ $= S^2 \times S^2$

$D($ " odd $) = S^2 \tilde{\times} S^2$

(3) $S^2 \tilde{\times} S^2 \cong \mathbb{C}P^2 \# (-\mathbb{C}P^2)$

(4) $(S^2 \times S^2) \# \mathbb{C}P^2 \cong (S^2 \tilde{\times} S^2) \# \mathbb{C}P^2$

Proof:

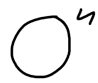
(1) * We cut any S^2-bundle over S^2 along an equator of the base to get two S^2-bundles over D^2

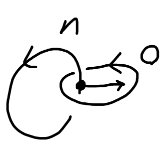
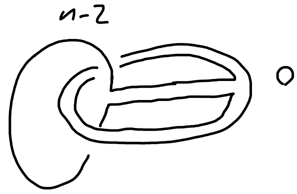
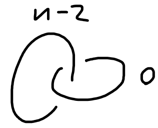
D_+ D_-

* $D^2 \simeq pt \Rightarrow \exists ! \, S^2$-bundle over D^2

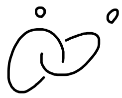
$\Rightarrow \# (S^2\text{-bundles over } S^2) = \# (\text{gluings})$ $SO(3) \simeq \mathrm{Diff}(S^2)$

$= \pi_1(SO(3)) = \pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$

(2) D^2-bundle over S^2 with Euler number n = 

$D(O^n) =$  $\overset{+}{=}$  $\cong$ 

$(lk = -1)$

n even $\Rightarrow$  $= S^2 \times S^2$ Q even

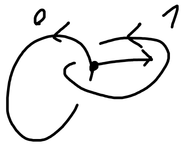
n odd $\Rightarrow$  $=: S^2 \tilde{\times} S^2$ Q odd

$\boxed{*}$ ⌈ $\boxed{Q}$ $\exists$ $E \overset{C^0}{\cong} S^2 \times S^2$ but $E \overset{C^\infty}{\not\cong} S^2 \times S^2$? ⌉

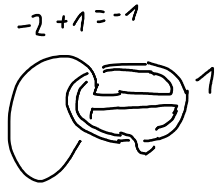
$$\begin{tikzcd}
E \arrow[r, "\overset{C^0}{\cong}", "f"'] \arrow[dr, "p\circ f"'] & W \arrow[d, "p \text{ smooth}"] & \supset & p^{-1}(U) \arrow[r, "\overset{C^\infty}{\cong}", "h"'] \arrow[d, "p"] & U \times S^2 \arrow[dl, "pr_1"] \\
& S^2 & \supset & U &
\end{tikzcd}$$

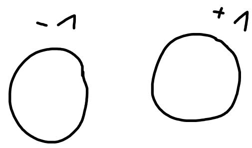
i.g. $p\circ f$ & $h \circ f$ are i.g. <u>NOT</u> smooth.

⌊ ⌋

(3) $S^2 \tilde{\times} S^2 =$ 

$(lk = +1)$

$\overset{\ominus}{\cong}$ 

$=$ 

$= (-\mathbb{CP}^2) \# \mathbb{CP}^2$

(4) C.2 & (1) & (3) ∎

Thm 4:

Let K_W be a Kirby diagram of W^4.

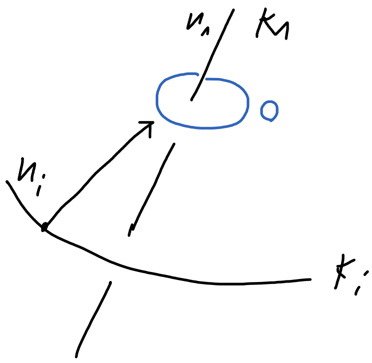
Let K_1, K_2 be attaching knots s.t.

$K_1 \subset \partial h_0$ & $K_2 = \mu_1$ with framing $= 0$

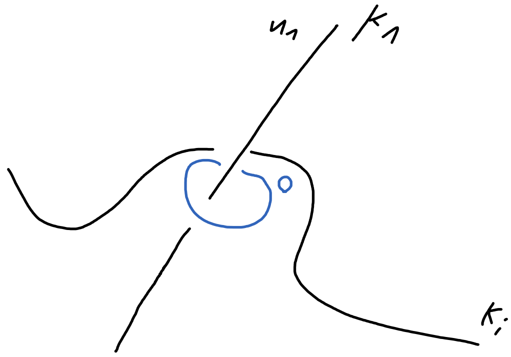
$$\Rightarrow \quad W \cong \begin{cases} X \# S^2 \times S^2 & \text{if } n_1 \in 2\mathbb{Z} \quad (\swarrow \text{ framing of } K_1) \\ X \# S^2 \tilde{\times} S^2 & \text{if } n_1 \in 2\mathbb{Z}+1 \end{cases}$$

where $X :=$ handle body without K_1 & K_2

Proof:

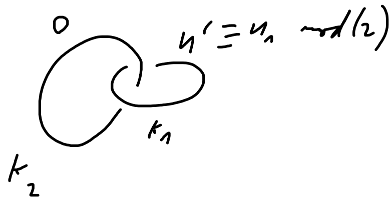


$\cong$



$K_i \neq K_1 \Rightarrow n_i' = n_i$

$K_i = K_1 \Rightarrow n_i' = n_i \pm 2$

after handle slides $\Rightarrow$ 

$\sqcup \quad K_X$ ∎

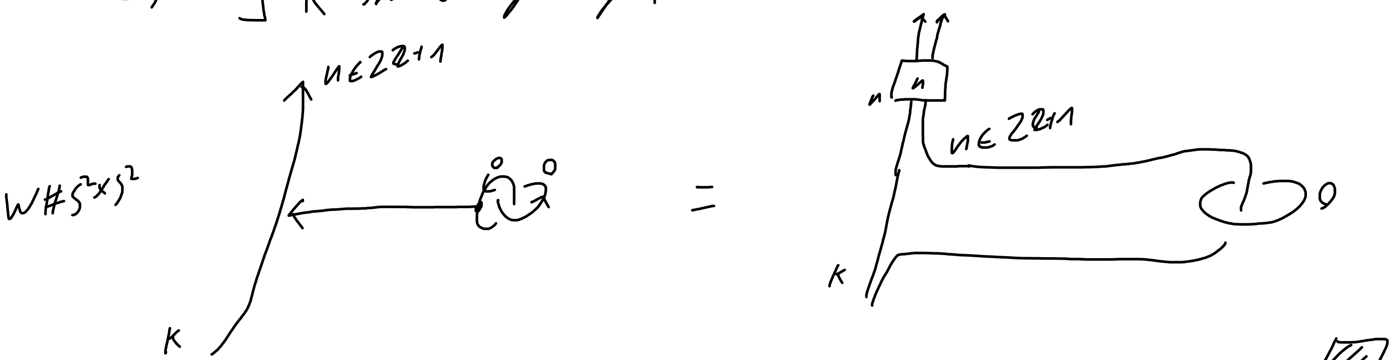
Corollary 5:

Let W^4 be without 1-handles & Q_W odd.

$$\Rightarrow \quad W \# S^2 \times S^2 \cong W \# S^2 \tilde{\times} S^2$$

Proof: Q_W odd

L.4.14 $\Rightarrow \exists K$ in Kirby diag of W with $n \in 2\mathbb{Z}+1$



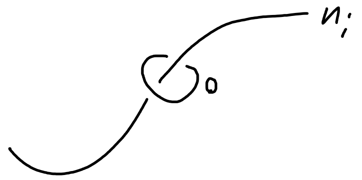
& Thm 4 $\blacksquare$

Corollary 6:

Let $W^4 = h_0 \cup \{2\text{-handles } h_2^i\}_{i=1}^m$

$$\Rightarrow \quad DW = \begin{cases} \#_m S^2 \times S^2 & \text{if } Q_W \text{ is even} \\ \#_m S^2 \tilde{\times} S^2 & \text{if } Q_W \text{ is odd} \end{cases}$$

Proof:

$DW =$ 

* Q_W even $\Leftrightarrow Q_W(\alpha,\alpha) \equiv 0 \mod (2) \Rightarrow$ all $n_i \in 2\mathbb{Z}$

$$\overset{\text{Thm 4}}{\Rightarrow} \quad DW = \#_m S^2 \times S^2$$

* Q_W odd $\Rightarrow$ one $n_i \in 2\mathbb{Z}+1$

$$\overset{\text{Thm 4}}{\Rightarrow} \quad DW = \#_{k_1} S^2 \tilde{\times} S^2 \ \#_{k_2} S^2 \times S^2 \qquad (k_1 + k_2 = m)$$

$$\overset{\text{C.5}}{=} \ \#_m S^2 \tilde{\times} S^2$$

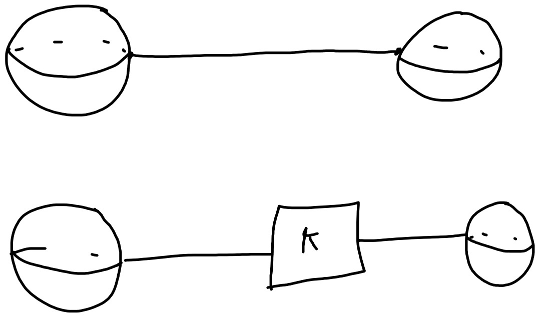
$\blacksquare$

5.2. HANDLE CANCELLATIONS

Recall: h_{k+1} & h_k cancel each other $\Leftrightarrow$ $a_{k+1} \pitchfork b_k = \{pt\}$

1-/2-cancelling pairs:

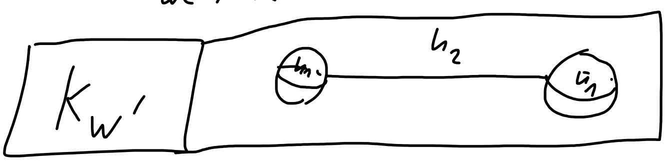
SHEET 3 EX 1

Ex:  $\cong \emptyset$

$\cong$

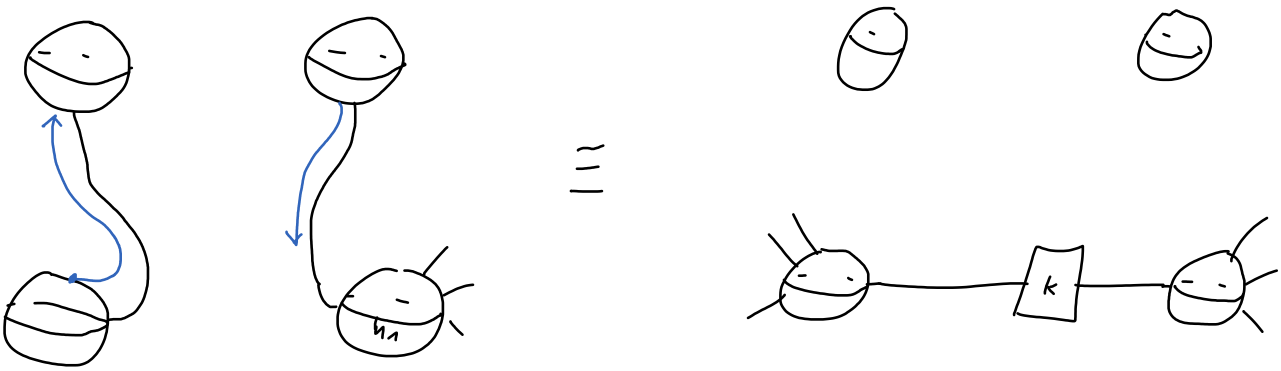
Lemma 7: Let $W_2 = h_0 \cup \{1\text{-handles}\} \cup \{2\text{-handles}\}$

h_1 & h_2 cancel each other $\Leftrightarrow$ After 1- & 2-handle slides we have

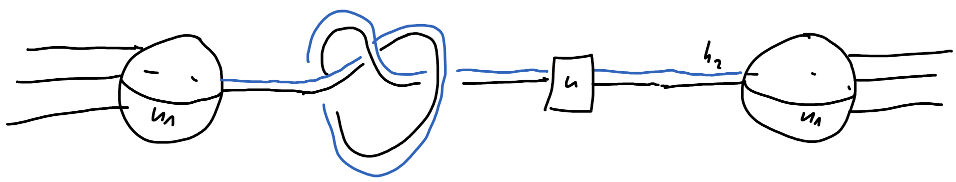


Proof: „$\Leftarrow$" ✓

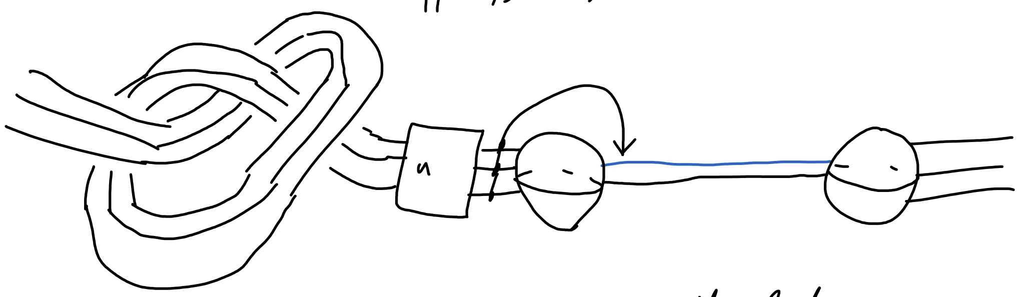
„$\Rightarrow$" ① After 1-handle slides we can assume that h_2 does NOT intersect any other 1-handle.



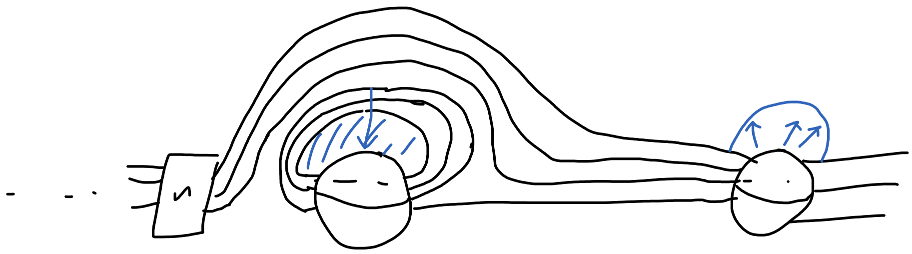
② After 2-handle slides we can assume that <u>NO</u> other 2-handle intersects h_1:



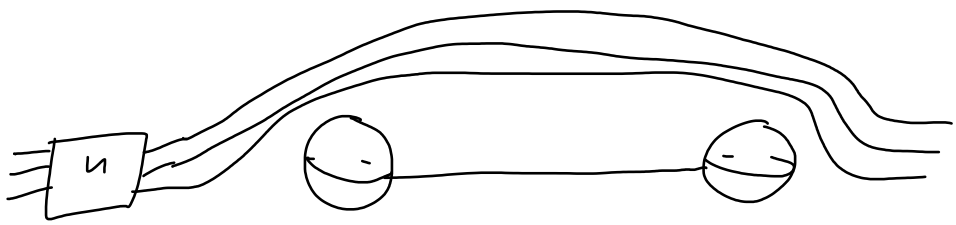
|| ISOTOPY



|| 2-handle slides



|| ISOTOPY

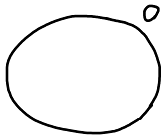


③ h_1 & h_2 do <u>NOT</u> intersect the other handles:

Claim follows from SHEET 3 EX1 $\blacksquare$

2-/3-CANCELLING PAIRS

Ex:



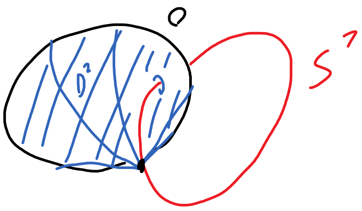
$\cup\, h_3 \;=\; \emptyset$

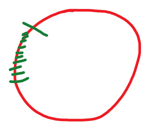
⌜

$\bigcirc^0 = S^2 \times D^2$

$\partial(\bigcirc^0) = \partial(S^2 \times D^2) = S^2 \times S^1$

$D^2 \cup$ copy of core of $h_2 = S^2$





$b_2 = \mu = pt \times S^1 \subset S^2 \times D^2$

h_3 attached via

$\varphi: \partial D^3 \times D^1 \hookrightarrow S^2 \times S^1 = \partial(S^2 \times D^2)$

$a_3 = S^2 \times pt$

$\Rightarrow a_3 \pitchfork b_2 = (S^2 \times pt) \pitchfork (pt \times S^1) = \{pt, pt\} \in S^2 \times S^2$

⌟

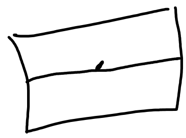
Lemma 8:

Let W^4 be closed.

h_2 & h_3 cancel each other $\Leftrightarrow$ we can isotope a_2 as a disjoint 0-framed unknot via 2- & 3-handle slide

Proof:

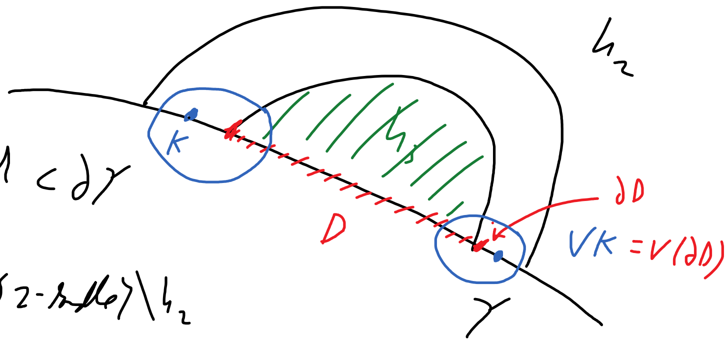
„$\Rightarrow$" Let h_2 & h_3 be a cancelling pair



w.l.o.g. $a_3 \cap h_2 = D^2 \times pt \subset D^2 \times \partial D^2 = \partial h_2$

$a_3 = \partial D^3 \times \{0\}$

$K = a_2$



$D = a_3 \setminus h_2$ is an embedded disk $\subset \partial Y$

where $Y = h_0 \cup \{1\text{-handles}\} \cup \{2\text{-handles}\} \setminus h_2$

$\Rightarrow \nu K = \nu(\partial D)$

$\Rightarrow$ we can isotope D in ∂Y s.t.

* K is isolated from the other 1- & 2-handles
* D is a Seifert disk of K & K is 0-framed

"$\Leftarrow$" Let h_2 be an isolated 0-framed unknot

$\Rightarrow \partial(Y \cup h_2) = \#_m S^1 \times S^2$

& h_2 determines one $S^1 \times S^2$-summand

$\exists!$ prime decomp of 3-mfd

$\Rightarrow \partial Y = \#_{m-1} S^1 \times S^2$

Laudenbach–Poénaru

$\Rightarrow Y \cup \{3\text{-handles}\}_{i=1}^{m-1} \cup h_4 \cong W$

∎

6. SURGERY

6.1. SURGERY & HANDLEBODIES

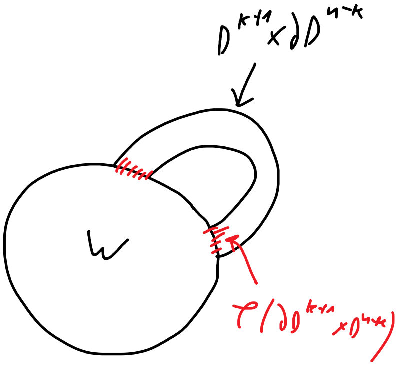
Let W^{n+1} be compact, smooth with $\partial W = M^n$

$$W^{n+1} \cup h_{k+1} = W^{n+1} \cup_\varphi D^{k+1} \times D^{n-k}$$

where $\varphi : \partial D^{k+1} \times D^{n-k} \hookrightarrow \partial W = M$

$M = \partial W$ changes to

$$M' := M \setminus \varphi(\partial D^{k+1} \times D^{n-k}) \cup D^{k+1} \times \partial D^{n-k}$$

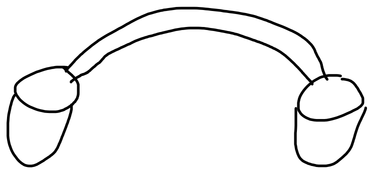


M' is obtained from M by <u>SURGERY</u> along $\varphi(\partial D^{k+1} \times D^{n-k})$

$$\boxed{\text{ATTACHING A } (k+1)\text{-HANDLE TO } W \;\hat{=}\; \text{PERFORMING A } k\text{-SURGERY ON } \partial W}$$

<u>Ex</u>: 0-SURGERY $\hat{=}$ ATTACHING A 1-HANDLE

$\Rightarrow$ $\#$ is a 0-SURGERY



<u>Corollary 1</u>:

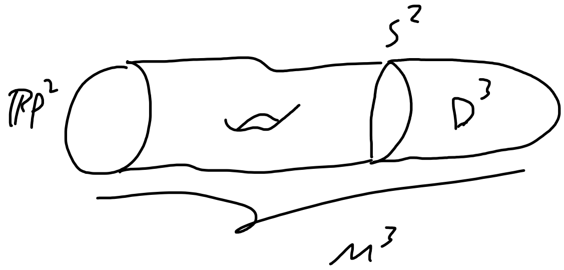
M' is obtained from M by a finite sequence of surgeries

$$\Leftrightarrow \quad \partial(\underbrace{I \times M \cup \text{handles}}_{=:W}) = -M \sqcup M'$$

$\Leftrightarrow$ $\exists$ <u>COBORDISM</u> W between M & M', i.e.

W compact or. mfd s.t. $\partial W = -M \sqcup M'$ ∎

Ex: $\mathbb{RP}^2$ is <u>NOT</u> cobordant to S^2

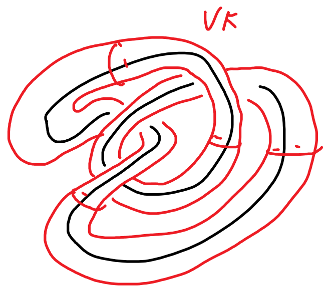


$1 = \chi(\partial M) = 2\,\chi(M^3) \neq 1$

6.2. DEHN SURGERY

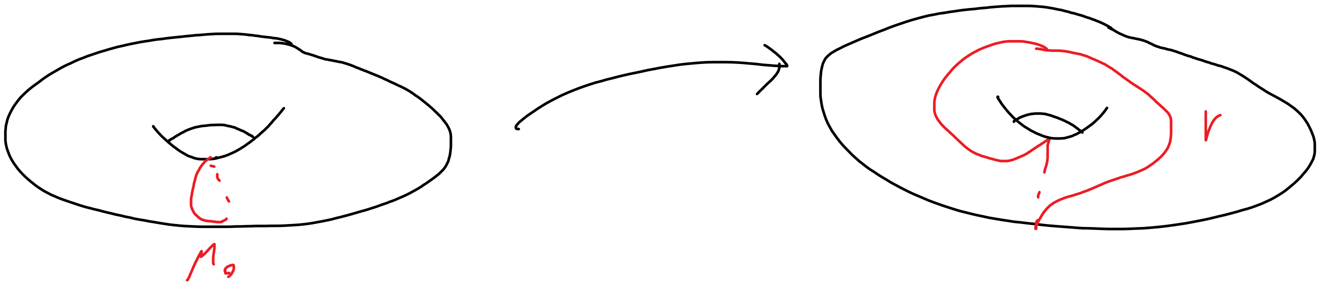
<u>Def</u>: Let M^3 oriented, closed 3-mfd, $K \subset M^3$ be a knot with tubular nbhd $VK \cong S^1 \times D^2$ & γ a non-trivial simple closed curve on ∂VK

<u>DEHN SURGERY</u> along K with <u>SLOPE</u> γ is



$$M_K(\gamma) := S^1 \times D^2 \quad \cup_\varphi \quad M \setminus \mathring{V}K$$

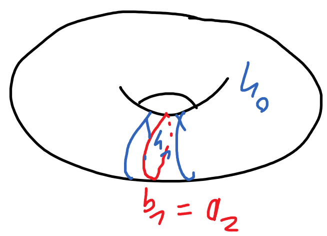
$$pt \times \partial D^2 =: \mu_0 \xmapsto[\cong]{\varphi} \gamma$$



Lemma 2:

$M_K(r)$ is independent of the choice of φ

Proof: $S^1 \times D^2 = h_0 \cup h_1$



$$M_K(r) = M \setminus \mathring{V}K \cup \underbrace{h_2 \cup h_3}_{\text{dual handle decomp of } S^1 \times D^2}$$

$$\varphi(\mu_0) = r = \text{attaching sphere of } h_2$$

attaching 3-handle is unique $\blacksquare$

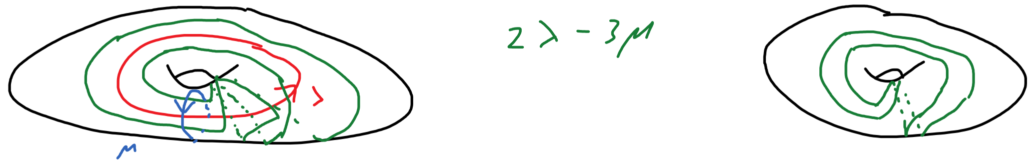
Ex: (0) $M_K(\mu_K) \cong M$

(1) $$L(p,q) = S^1 \times D^2 \cup_\varphi S^1 \times D^2$$
$$\mu_0 \longmapsto q\mu_1 - p\lambda_1$$
$$= S^1 \times D^2 \cup_\varphi S^3 \setminus \mathring{V}U$$
$$= S^3_U(q\mu_1 - p\lambda_1)$$

$\boxed{Q}$ How to describe r?

Let λ be a LONGITUDE on ∂VK, i.e. a parallel copy of K on ∂VK

$\Rightarrow \forall$ slope r $\exists!$ (p,q) coprime s.t. $r = p\mu + q\lambda$



For a given λ we express a slope r by its

SURGERY COEFFICIENT $r = p/q \in \mathbb{Q}$

For $K \subset S^3$ we choose λ = Seifert longitude / framing

Corollary 3

INTEGER SURGERY (i.e. $r \in \mathbb{Z}$) corresponds to attaching a 2-handle to $M \times I$ (or any W with $\partial W \supset M$)

Ex: $\bigcirc^{0} = S^1 \times S^2$

Proof:

$r \in \mathbb{Z}$, we glue via:

$$\mu_0 \longmapsto r\mu + \lambda = \lambda' = \text{framing of } K \qquad \square$$

Ex: (1) $L(p,q) = \bigcirc^{-p/q}$

$L(n,1) = \bigcirc^{-n}$

(2) $\bigcirc^{\pm 1} = S^3$ $\qquad \bigcirc^{\pm 1/n} = S^3$

(3) $T^3 \neq S^3_K(r)$ $\qquad r = p/q$

$\Big[$ $H_1(S^3_K(r)) = \langle \mu_K \mid p\mu_K = 0 \rangle \cong \mathbb{Z}_p$

but $H_1(T^3) \cong \mathbb{Z}^3$ $\Big]$

THM 4 (LICKORISH – WALLACE)

$\forall\, M^3$ closed, oriented, con. $\exists$ **LINK** $L \subset S^3$ s.t.

$$M \cong S^3_L(n_i) \quad \text{with } n_i \in \mathbb{Z}$$

THM 5 (ROHLIN)

$\forall\, M^3$ closed, oriented, con. $\exists$ compact 4-mfd W with $\pi_1(W) = 1$ s.t.

$$\partial W \cong M$$

CLAIM: THM 4 $\Leftrightarrow$ THM 5

„$\Rightarrow$"
- start with $\partial h_0 = \partial D^4 = S^3$
- attach 2-handles that correspond to the surgery
- $W = h_0 \cup \{2\text{-handles}\}$ with $\partial W = M$

„$\Leftarrow$" let W^4 with $\partial W = M$

PROBLEM: W has i.g. 1- & 3-handles

$W_1 = \sqcup_k S^1 \times D^3$

$W_1' := \bigcirc^0 \; \dots \; \bigcirc^0 = \sqcup\, D^2 \times S^2$ with $\partial W_1' = \partial W_1$

replace W_1 by W_1' $\longrightarrow$ NO 1-handles

by duality $\longrightarrow$ NO 3-handles $\blacksquare$

Proof sketch of THM 4

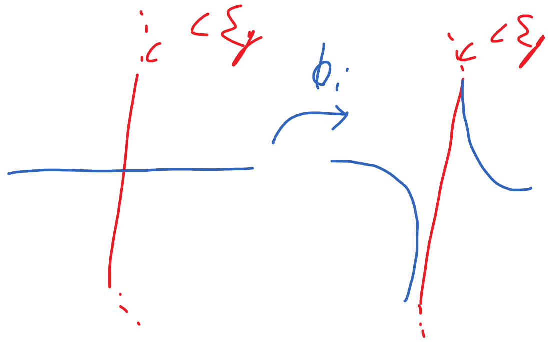
Let $M = H_1 \cup_\psi H_2$ be a Heegaard splitting

$$\psi : \Sigma_g \xrightarrow{\cong} \Sigma_g$$

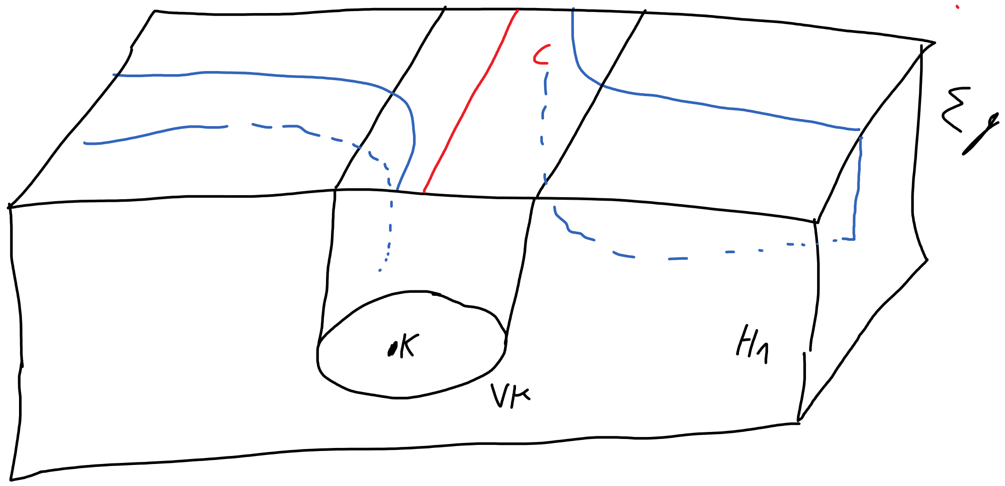
Let $S^3 = H_1 \cup_\varphi H_2$ with $\varphi : \Sigma_g \xrightarrow{\cong} \Sigma_g$

be the genus-g-Heegaard splitting of S^3

We get M from S^3 by cutting along Σ_g and re-gluing via a diffeo f of Σ_g



$\Rightarrow$ $f = \prod_{i=1}^{n} \phi_i$ with ϕ_i = Dehn-twist



$$\mu_0 \xmapsto{\varphi} \pm \mu + \lambda_{\Sigma_g}$$

$\square$

$\hat{=}$ integer surgery along K

THM 6 (KAPLAN)

$\forall M^3$ compact, oriented $\exists k \in \mathbb{N}_0$

(1) $\exists\ M^3 \hookrightarrow \#_k S^2 \tilde{\times} S^2$

(2) $\exists\ M^3 \hookrightarrow \#_k S^2 \times S^2 \hookrightarrow \mathbb{R}^5 \subset S^5$

Proof: w.l.o.g. M closed (if NOT, consider DM)

Let $W^4 = h_0 \cup \{2\text{-handles}\}$ with $\partial W = M$ (THM 4)

C.S.6.
$$\Rightarrow \quad DW = \begin{cases} \#_k S^2 \times S^2 & ; \text{ if } Q_W \text{ even} \Leftrightarrow \forall n_i \text{ even} \\ \#_k S^2 \tilde{\times} S^2 & ; \text{ if } Q_W \text{ odd} \Leftrightarrow \exists n_i \text{ odd} \end{cases}$$

* If Q_W is even $\Rightarrow Q_{W \# \mathbb{CP}^2}$ is odd $\Rightarrow$ (1)

* $\# S^2 \times S^2 \hookrightarrow S^5$ as $\# S^7 \times S^7 \hookrightarrow S^3$:

$$\left\ulcorner S^5 = \partial D^6 = \partial(D^3 \times D^3) = S^2 \times D^3 \cup \underbrace{D^3 \times S^2}_{\subset\, \partial D^3 \times S^2 = S^2 \times S^2} \right\lrcorner$$

THM 6 follows from:

Lemma 7:

$\forall M^3$ (closed, or., con.) $\exists$ surgery description of M s.t.

all coeff $n_i \in 2\mathbb{Z}$

Proof: at the end of this section $\boxed{}$

Corollary 8:

$\forall$ M^3 closed, or., conn.

(1) $\exists$ W^4 SPIN (i.e. $w_2(TW) = w_2(W) = 0$) s.t. $\partial W = M$

(2) $TM \cong M \times \mathbb{R}^3$, i.e. M is PARALLELIZABLE.

Proof idea:

(1) $\exists\, W$ s.t. $\partial W = M$ & $W = h_0 \cup \{h_2^i\}$ with $n_i \in 2\mathbb{Z}$

$\Rightarrow$ Q_W is even $\Rightarrow$ $w_2(W) = 0$

(2) start with trivialization of $S^3 \subset \mathbb{R}^4 \cong \mathbb{H}$ (QUATERNIONS) given by

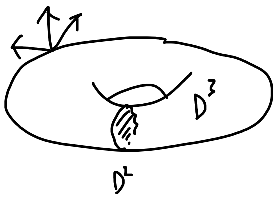
$$X_1(p) = ip$$
$$X_2(p) = jp$$
$$X_3(p) = kp$$

* let $L(n_i)$ be a framed surgery link of M

let (X_1, X_2, X_3) be the framing of S^3 on $\partial \nu L_i \cong \partial S^1 \times D^2$

(X_1, X_2, X_3) extends to $S^1 \times D^2$ $\Leftrightarrow$ $f : \partial(S^1 \times D^2) \longrightarrow SO(3)$

extends to $S^1 \times D^2$



$\Leftrightarrow$ $\tilde{f} : pt \times \partial D^2 \longrightarrow SO(3)^{=\mathbb{R}P^3}$ extends

$\hat{f} : \partial D^3 \longrightarrow SO(3)^{=\mathbb{R}P^3}$ extends

$\Leftrightarrow$ $0 = [\tilde{f}] \in \pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$

$0 = [\hat{f}] \in \pi_2(\mathbb{R}P^3) = 0$

$\Leftrightarrow$ $n_i \in 2\mathbb{Z}$

computation

∎

6.3. KIRBY'S THM:

Thm 9 (KIRBY)

Let $L_1, L_2 \subset S^3$ be framed links representing compact W_1, W_2.

$\partial W_1 \overset{C^\infty}{\cong} \partial W_2 \iff L_2$ can be obtained from L_1 by finitely many 2-handle slides & <u>BLOW UPS / DOWNS</u>, i.e.

$$\begin{tikzcd} L \arrow[r, bend left, "\text{BLOW UP}"] & L \sqcup \bigcirc^{\pm 1} \arrow[l, bend left, "\text{BLOW DOWN}"] \end{tikzcd}$$

$$W \longrightarrow W \# \pm \mathbb{C}P^2$$

Proof sketch: „$\Leftarrow$" ✓

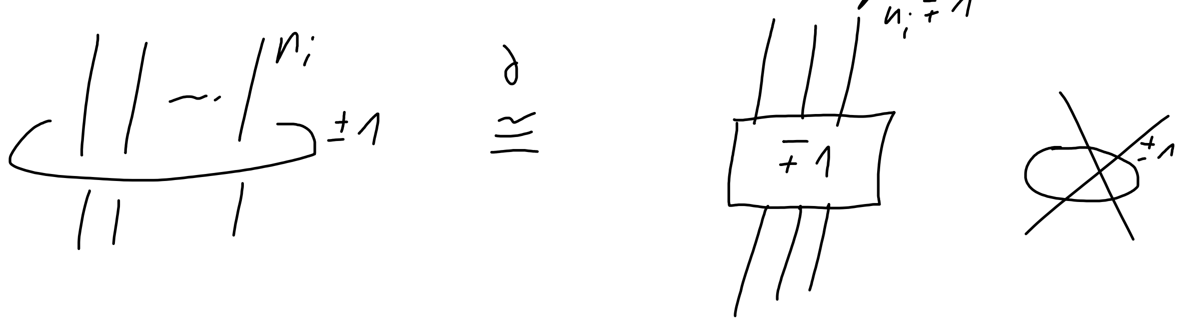
„$\Rightarrow$"

$$\begin{tikzcd} L_1 \arrow[r] \arrow[d, "\substack{\text{2-handle slide}\\ \text{blowups/downs}}"] & \text{Heegaard splitting of } \partial W_1 \arrow[d, "\substack{\text{stab/destab}\\ \&\\ \text{handle slides}}", "\substack{\text{REIDEMEISTER}\\ \text{-SINGER}}"'] \\ L_2 \arrow[r] & \text{Heegaard splitting of } \partial W_2 \end{tikzcd}$$

∎

Corollary 10

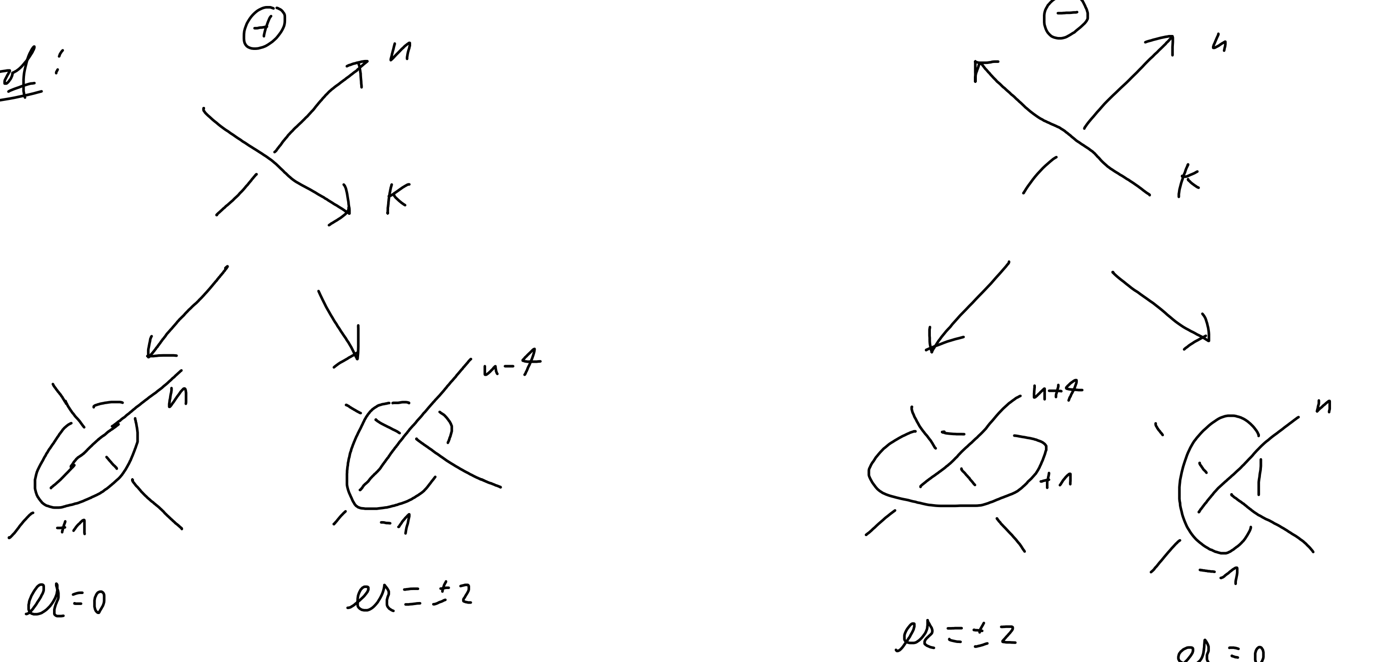
$\partial W_1 \cong \partial W_2 \iff L_2$ can be obtained from L_1 by



BLOW UP / DOWN

Lemma 11: We can change crossings by blow ups.

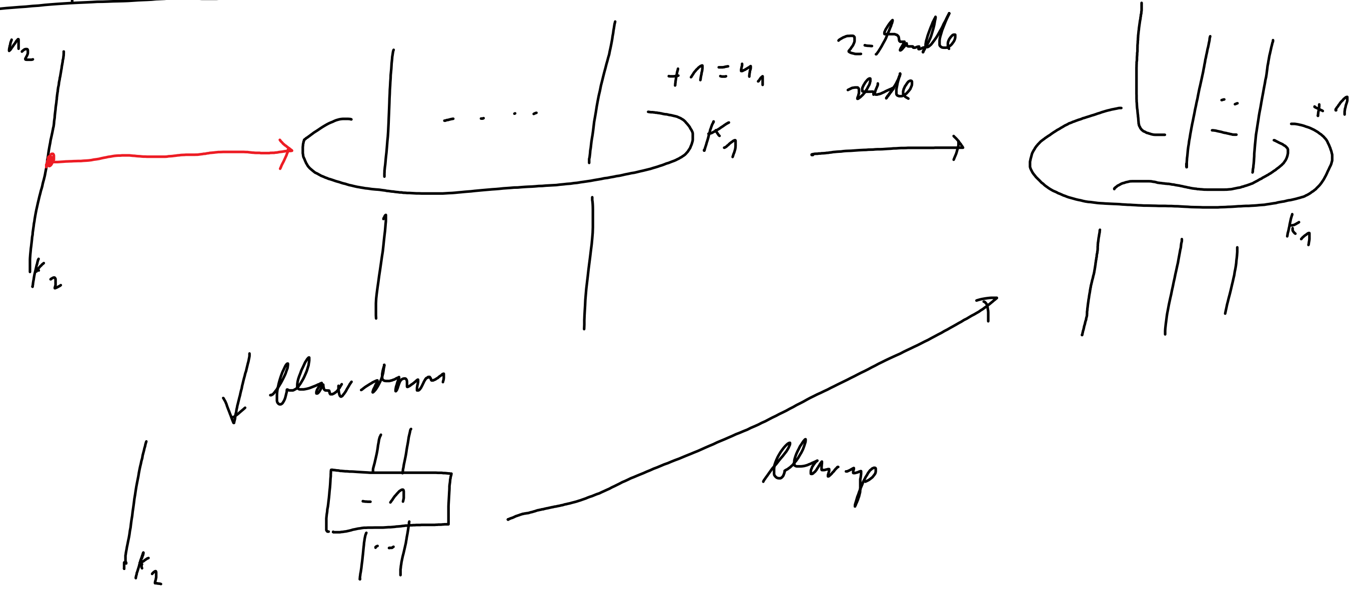
Proof:



∎

Proof of C.10

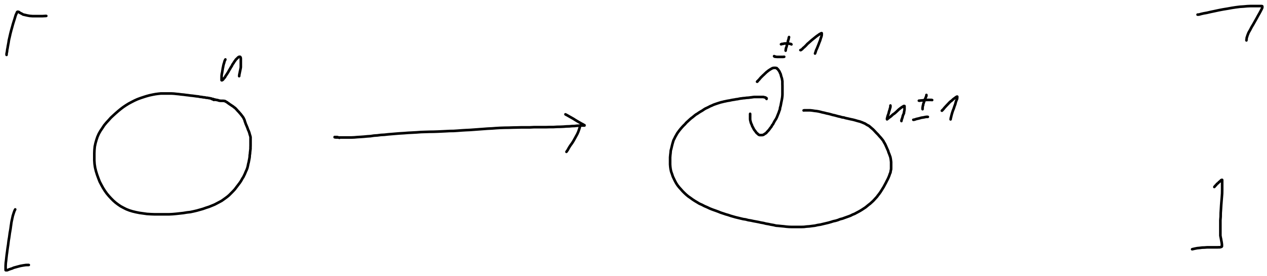
(1)



Let K_1 be arbitrary

② L.11 $\Rightarrow$ After blow ups: K_1 = unknot

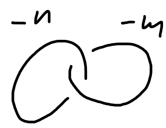
③ After blow ups: framing $n_1 = +1$



④ Perform the 2-handle slide as in ①

⑤ Reverse all blow ups from ② & ③ by blow downs. ∎

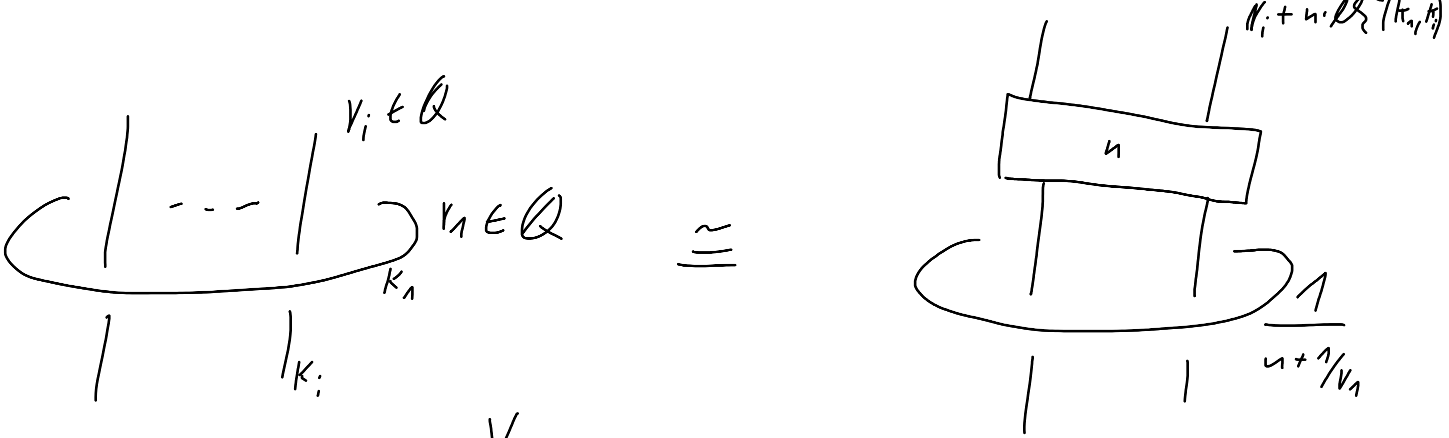
<u>Ex:</u>



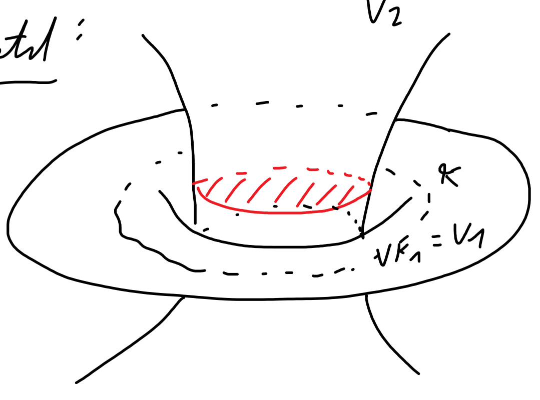
$= L(4n-1, 4) = L(4n-1, 4)$

this is easy to see via rat. coeff.

<u>Lemma 12</u> (ROLFSEN TWIST)

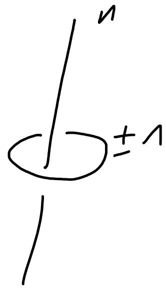


<u>Proof sketch:</u>

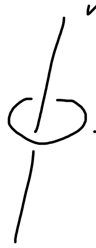


* Cut V_2 along a meridional disk
* perform n- full twists
* re-glue
* check the framings

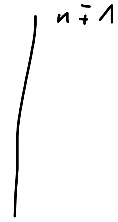
Ex: Blow up „C" Rolfsen twist



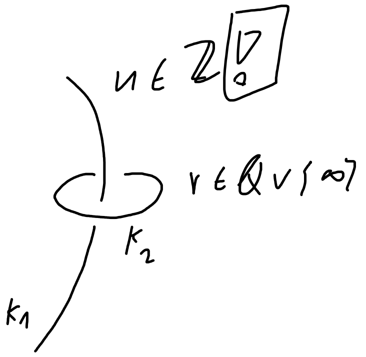
(∓ 1)-RT $\cong$



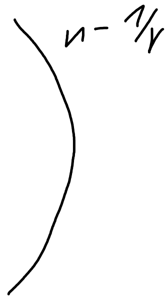
$$\frac{1}{\mp 1 + \frac{1}{\pm 1}} = \infty = \frac{1}{0} \hat{=} 1\cdot\mu + 0\cdot\lambda =$$



Lemma 13:



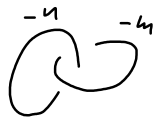
$\cong$



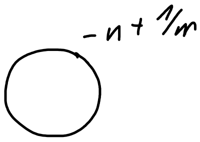
* we can express a rational twist via Rolfsen twists.

Proof: similar as L.12 & L.10 ∎

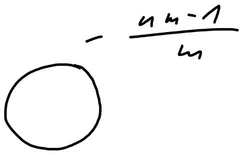
Ex: *



s.D. $\cong$

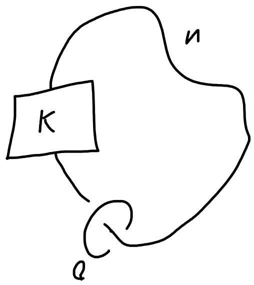


=

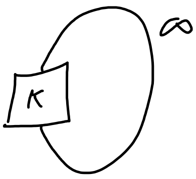


$= L(nm-1, m)$

*



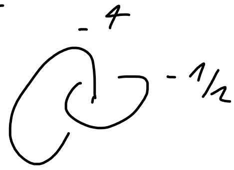
s.D. =



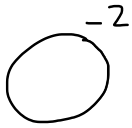
$= \emptyset = S^3$

(c.f. double DW_2)

* WARNING



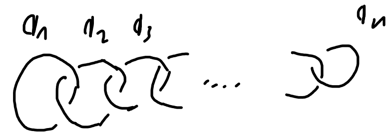
s.D. $\cong$



$= L(2,1) = \mathbb{R}P^3$

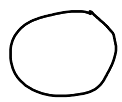
$\not\cong$

$= S^3$

* 

$a_i \in \mathbb{Z}$

$\| \text{SD}$



$$r = a_n - \cfrac{1}{a_{n-1} - \cdots - \cfrac{1}{a_1}}$$

$$= L(p,q)$$

* Conversely, we can deform any rational surgery diagram into an integer surgery diagram

<u>Corollary 14:</u>

Let L_1 & $L_2 \subset S^3$ be rational surgery descriptions of M_1 & M_2

$M_1 \cong M_2 \iff L_2$ can be obtained from L_1 by finitely many RT & introducing/removing ∞-comp.

<u>Proof:</u> * Transform L_1 & L_2 via slam dunks into integer surgery diag

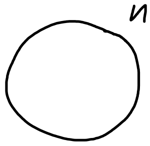
* use C.10 & C.13 $\blacksquare$

Lemma 7:

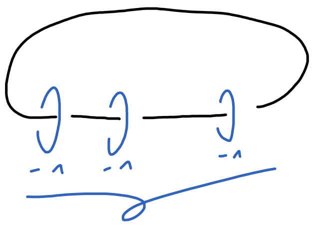
$\forall M^3 \ \exists$ surgery diag $L \subset S^3$ with $n_i \in \mathbb{Z}$

Ex:

(1)

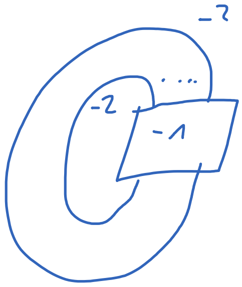


Blow ups $=$



$(n-1)$-times

Blow down $=$

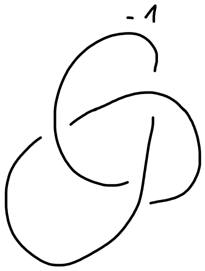


$Q = (n)$

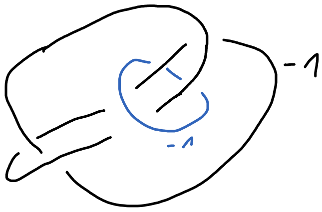
$$Q = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & -1 & & 0 \\ \vdots & & \ddots & \\ 1 & 0 & & -1 \end{pmatrix}$$

$$Q = \begin{pmatrix} -2 & & * \\ & \ddots & \\ * & & -2 \end{pmatrix}$$

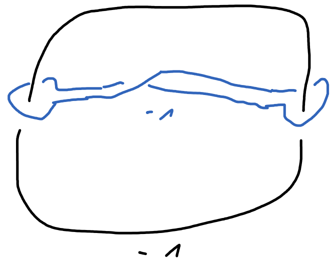
(2)



Blow up $\cong$



$\cong$

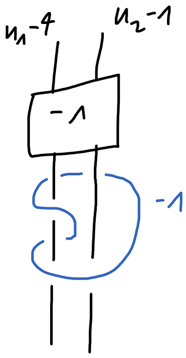


$Q = (-1)$

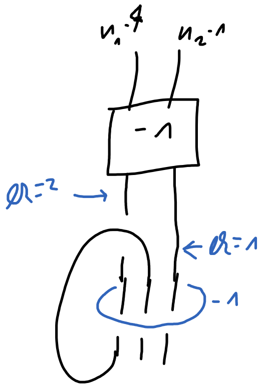
$Q = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$Q = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

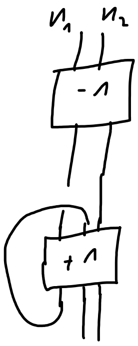
K-Blow up/Down



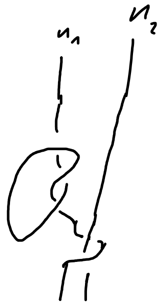
Isotopy $\cong$



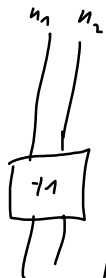
L. 11 $\cong$



$\cong$



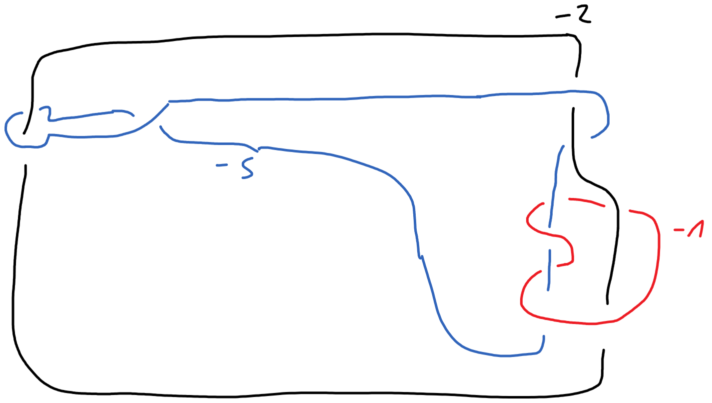
$=$



$\cong$

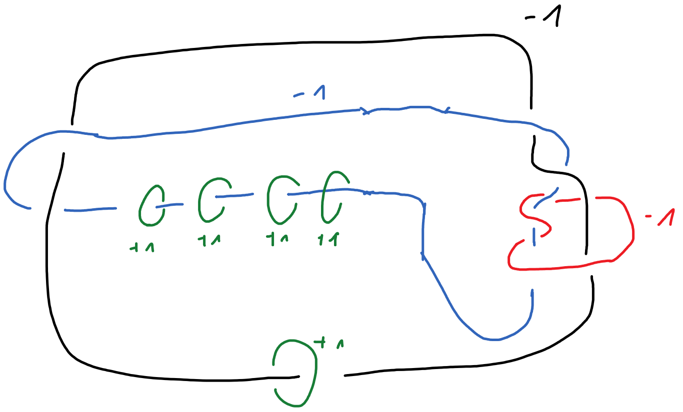
K BLOW UP

$=$



$$Q = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -5 & 2 \\ 1 & 2 & -1 \end{pmatrix}$$

$=$



$$Q = \begin{pmatrix} -1 & 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 0 & 1 & \cdots & 1 \\ 1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & +1 & & & \\ 0 & 1 & & & \ddots & 0 & \\ \vdots & \vdots & \vdots & & 0 & \ddots & \\ 0 & 1 & 0 & & & & +1 \end{pmatrix}$$

BLOW DOWN BLUE & BLACK

$$= \quad Q = \begin{pmatrix} -1+1^2+2^2=4 & & \ast \\ & 1+1^2=2 & \\ \ast & & \ddots \;\; 2 \end{pmatrix} \text{ even}$$

Proof of L.7

Let $L = L_1 \cup \dots \cup L_k$ be an integer surgery diag. of M

$$Q = \begin{pmatrix} n_1 & & lk_{ij} \\ & \ddots & \\ lk_{ij} & & n_k \end{pmatrix} \quad \text{the linking matrix}$$

w.l.o.g. let $n_1, \dots, n_i$ even & $n_{i+1}, \dots, n_k$ odd

(1) 2-handle slides of n_j over n_k for $j = i+1, \dots, k-1$ yields

$$(n_j' = n_j + n_k \pm 2 lk)$$

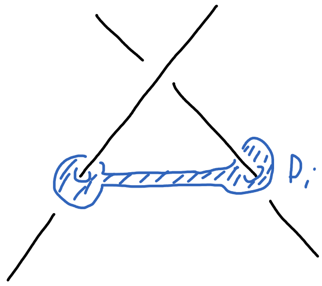
$$Q = \begin{pmatrix} n_1 & & lk_{ij} \\ & \ddots & \\ lk_{ij} & & n_k \end{pmatrix} \quad \text{with} \quad n_1, \dots, n_{k-1} \text{ even } \& \ n_k \text{ odd}$$

② Perform blow ups with lk $=0$ s.t. $L_K = K_0 =$ unknot.

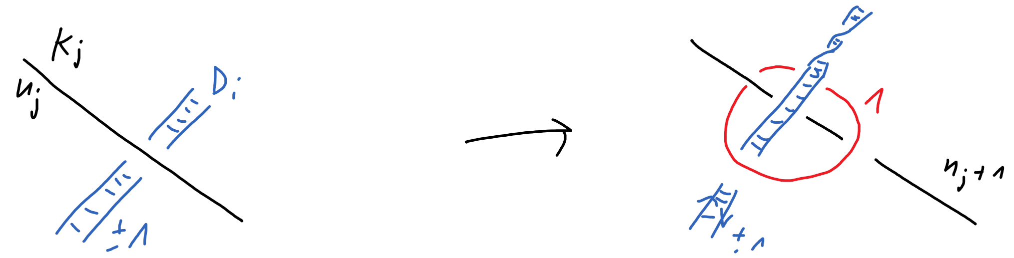
we get (± 1)-framed unknots $K_1, \dots, K_r$

$$Q = \begin{pmatrix} \begin{matrix} n_1 & lk \\ lk & n_k \end{matrix} & 0 \\ 0 & \begin{matrix} \pm 1 & 0 \\ 0 & \pm 1 \end{matrix} \end{pmatrix}$$

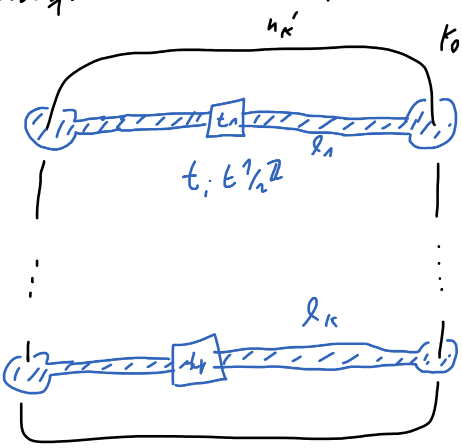
$L_1 — L_K \qquad k_1 — k_r$

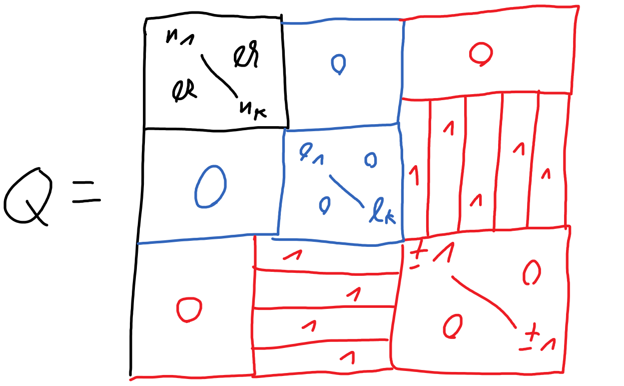


③ Add blow ups of the form:



to transform $K_0 \cup \dots \cup K_r$ into:

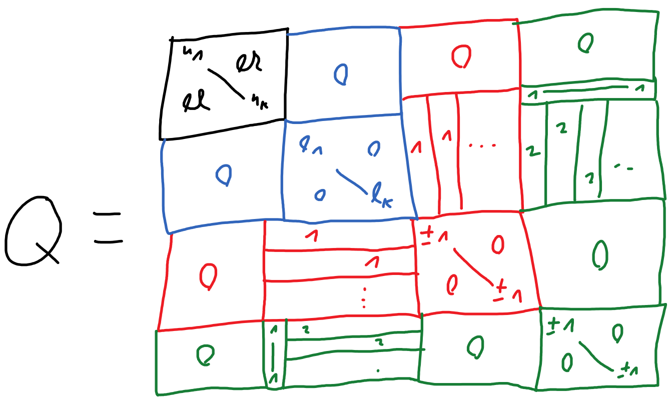




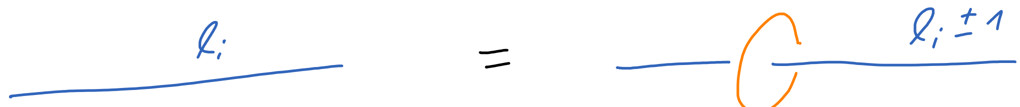
all entries $=0$ except for one 1

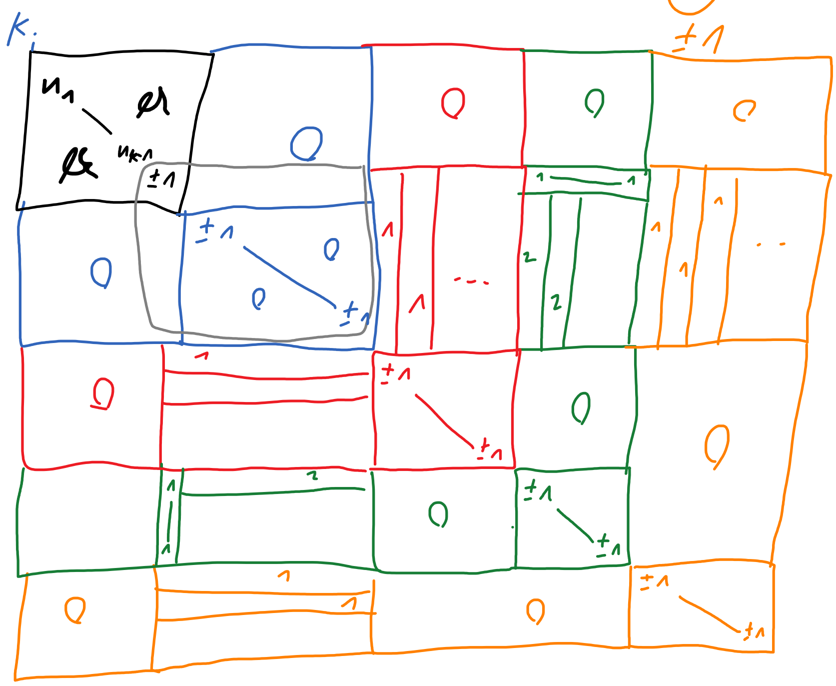
④ Add k-blow ups s.t.

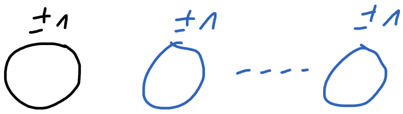
$K_0 \cup \dots \cup K_r$ is an unlink



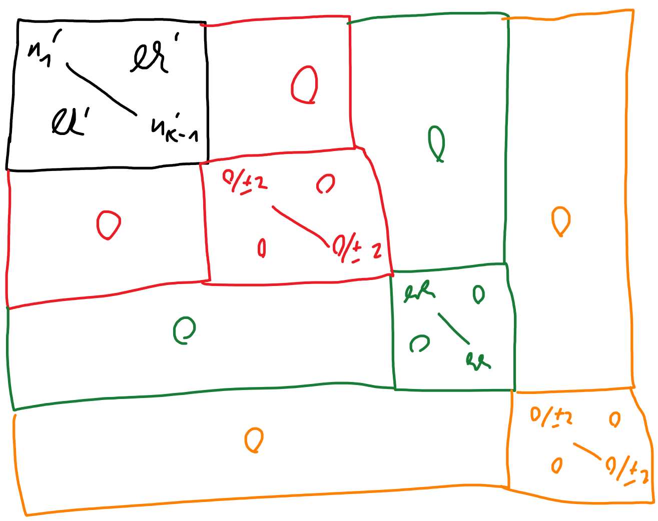
⑤ Blow up along μ_{K_i}'s s.t. $n_K, \ell_1, \dots, \ell_r = \pm 1$



$Q =$ 

⑥ $K_0 \cup \dots \cup K_r =$  unlinked

We blow down $K_0 \cup \dots \cup K_r$:

$Q =$ 

⑦ if $n_1', \dots, n_{k-1}' \notin 2\mathbb{Z}$ start at ① to further reduce k ▨

7. STABILIZATION THMS FOR SIMPLY-CONNECTED 4-MFDS:

THM 1:
Let W_1, W_2 closed connected smooth 4-mfd with $\pi_1 = 1$

$\Rightarrow \exists\, k_1, l_1, k_2, l_2 \in \mathbb{N}_0$:

$$W_1 \#_{k_1} \mathbb{CP}^2 \#_{l_1} -\mathbb{CP}^2 \overset{C^\infty}{\cong} W_2 \#_{k_2} \mathbb{CP}^2 \#_{l_2} -\mathbb{CP}^2$$

Proof: (1) Assume: W_1 & W_2 admit handle decomp WITHOUT 1-handles

$\Rightarrow W_i$ describe by Kirby diag $L_i \subset S^3$

* We see $L_i \subset S^3$ as a surgery diagram of

$$M_i = \partial(W_{i,2}) = \#_{m_i} S^1 \times S^2 \qquad (W_i \text{ closed})$$

* Add $\sqcup\, \bigcirc^0$ to L_i s.t.

$$M_1 = \partial(W_{1,2}) = \#_m S^1 \times S^2 = \partial(W_{2,2}) = M_2$$

[Adding $\sqcup\, \bigcirc^0$ $\hat{=}$ introducing cancelling 2-/3- handle pair
& thus does NOT change W_i]

* THM 6.9. (KIRBY)
$\Rightarrow L_2$ can be obtained from L_1 by 2-handle slides & adding/deleting $\bigcirc^{\pm 1}$

* 2-handle slides do NOT change W_i

* adding $\bigcirc^{\pm 1}$ changes W_i to $W_i \# \pm\mathbb{CP}^2$ ∎

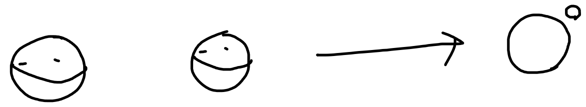
Lemma 2:

Let W^4 smooth, closed, connected with $\pi_1 = 1$

Replacing $W_1 = \natural_k S^1\times D^3$ by $\natural_k D^2 \times S^2$

$\boxed{\partial: \# S^1\times S^2}$

$\boxed{\tilde{W} = W \setminus \natural_k S^1\times D^3 \cup \natural_k D^2\times S^2}$

(i.e. 

changes W to $W \#_k$ S^2-bundle over S^2

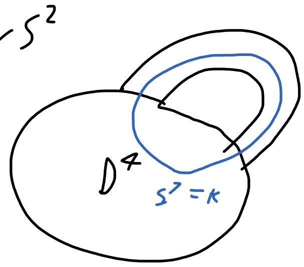
- S^2-bundles over S^2 = $S^2\times S^2$ or $S^2 \tilde{\times} S^2$
- $S^2 \tilde{\times} S^2 = \mathbb{CP}^2 \# -\mathbb{CP}^2$
- $S^2\times S^2 \# \mathbb{CP}^2 = \mathbb{CP}^2 \# \mathbb{CP}^2 \# -\mathbb{CP}^2$ this finishes the proof of T.1.

Proof of L.2:

$\bigcirc^4$ = S^2-bundles over S^2

(1) Surgery along $S^1\times\{0\} \subset S^1\times D^3$ yields $D^2\times S^2$

(2) Surgery along $S^1 = \partial D^2 \subset S^4$ yields an S^2-bundle over S^2



- Let $K = S^1 \subset W^4$ be the core of 1-handle closed to an S^1

$\Rightarrow$ W with 1-handle replaced by $D^2\times S^2$

$\overset{(1)}{\cong}$ surgery along $K \subset W$

$\pi_1 = 1 \Rightarrow K$ homotopic to $S^1 = \partial D^2$

dim $\geq 4 \Rightarrow K$ isotopic to $S^1 = \partial D^2$

$\cong$ surgery along $S^1 = \partial D^2 \subset S^4$ in $W = W \# S^4$

$\overset{(2)}{\cong}$ $W \# S^2$-bundle over S^2 ▨

Corollary 3:

Let W closed, connected, smooth with $\pi_1 = 1$

$$\Rightarrow \exists n \in \mathbb{N}_0 : W \#_n \mathbb{C}P^2 \#_n -\mathbb{C}P^2 \overset{C^\infty}{\cong} \#_{b_2^+(W)+n} \mathbb{C}P^2 \#_{b_2^-(W)+n} -\mathbb{C}P^2$$

Proof: THM 1 $\Rightarrow \exists n$ s.t.

$$W \#_n \mathbb{C}P^2 \#_n (-\mathbb{C}P^2) \overset{C^\infty}{\cong} \#_{b^+ + n} \mathbb{C}P^2 \#_{b^- + n} -\mathbb{C}P^2$$

$$\Rightarrow Q_{(\dots)} = Q_{(\dots)}$$

$$\Rightarrow b^+ = b_2^+(W) \ \& \ b^- = b_2^-(W)$$ $\blacksquare$

Corollary 4:

Let W_1, W_2 be closed, con, smooth with $\pi_1 = 1$.

$$W_1 \overset{C^0}{\cong} W_2 \Rightarrow \exists K \in \mathbb{N}_0 : W_1 \#_K S^2 \tilde{\times} S^2 \cong W_2 \#_K S^2 \tilde{\times} S^2$$

Proof:

THM 1 $\Rightarrow W_1 \#_{k_1} \mathbb{C}P^2 \#_{l_1} -\mathbb{C}P^2 \overset{C^\infty}{\cong} W_2 \#_{k_2} \mathbb{C}P^2 \#_{l_2} -\mathbb{C}P^2$

$$W_1 \overset{C^0}{\cong} W_2 \Rightarrow Q_{W_1} = Q_{W_2} \Rightarrow k_1 = k_2 \ \& \ l_1 = l_2$$

After adding more $\pm \mathbb{C}P^2$ $\Rightarrow k_1 = k_2 = l_1 = l_2$

& from $S^2 \tilde{\times} S^2 \overset{C^\infty}{\cong} \mathbb{C}P^2 \# -\mathbb{C}P^2$ we get the claim $\blacksquare$

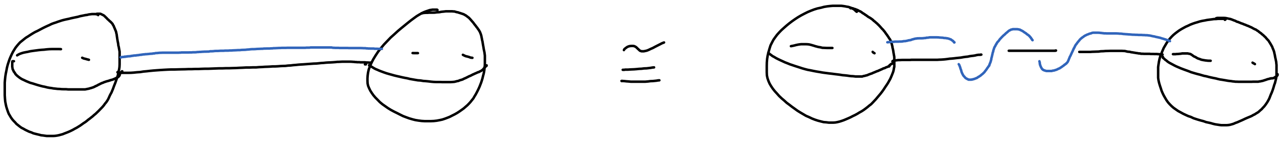
$\boxed{Q}$ How can we prove Wall's thm via Kirby calculus?

Remark: All these results are wrong if $\pi_1 \neq 1$

$\Rightarrow$ Kirby's thm for in general 3-mfds WRONG

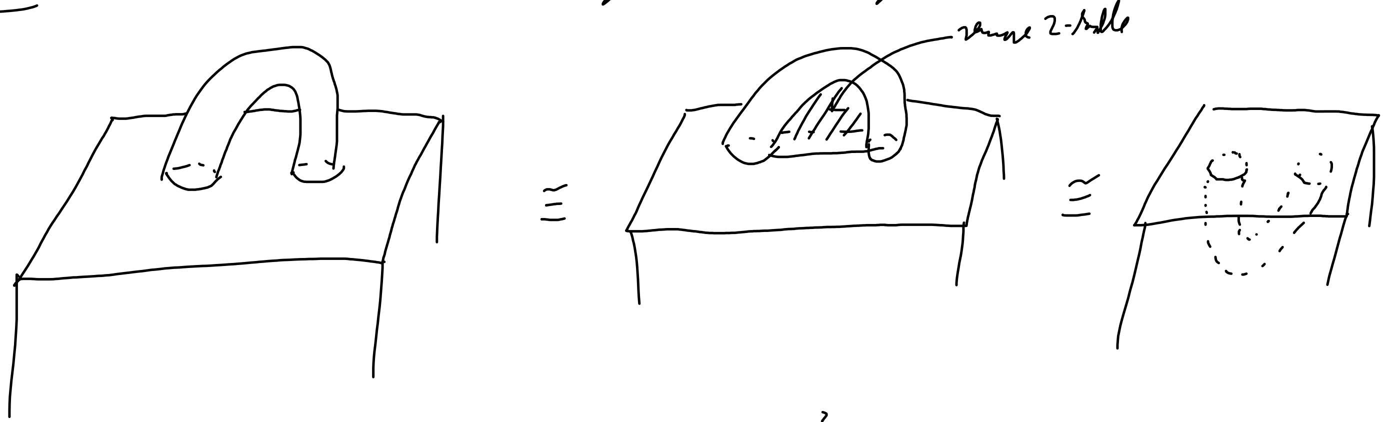
8. DOTTED CIRCLE NOTATION OF 1-HANDLES

PROBLEM: NO „0-framing" for NOT-nullhomologous knots

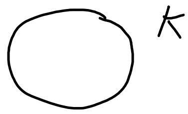


$\rightarrow$ new description of 1-handles

IDEA: attach 1-handle $\hat{=}$ removing a cancelling 2-handle

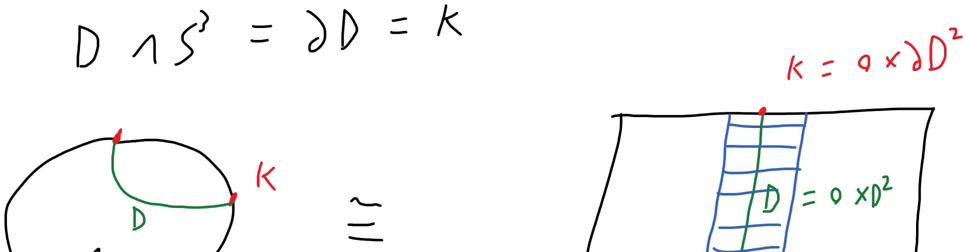


* $D^4 \cup h_1 = D^1 \times D^3 \cup D^1 \times D^3 = S^1 \times D^3$
* let $K \subset S^3 = \partial D^4 = \partial h_0$ be an unknot



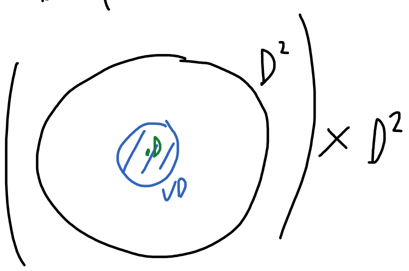
let $D \subset D^4$ be a Seifert disk of K pushed into D^4, i.e.

$$D \cap S^3 = \partial D = K$$



$D^2 \times D^2 = D^4$

$$\Rightarrow D^4 \setminus \nu D \cong D^2 \times D^2 \setminus \nu D \cong I \times S^1 \times D^2 \cong S^1 \times D^3$$

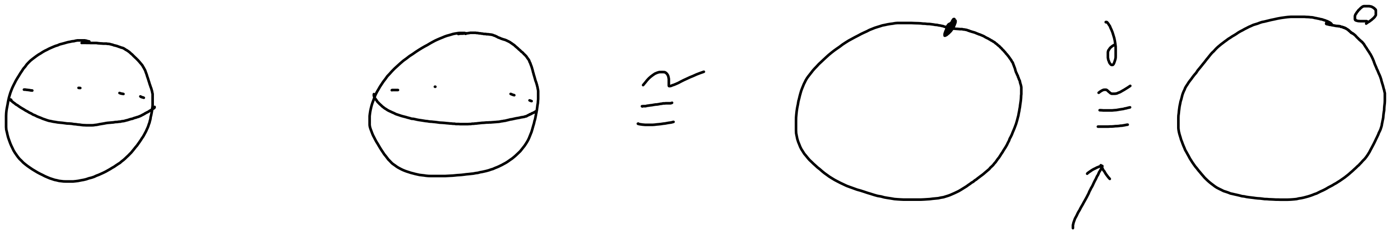


i.e. attaching 1-handle

$\hat{=}$ removing νD

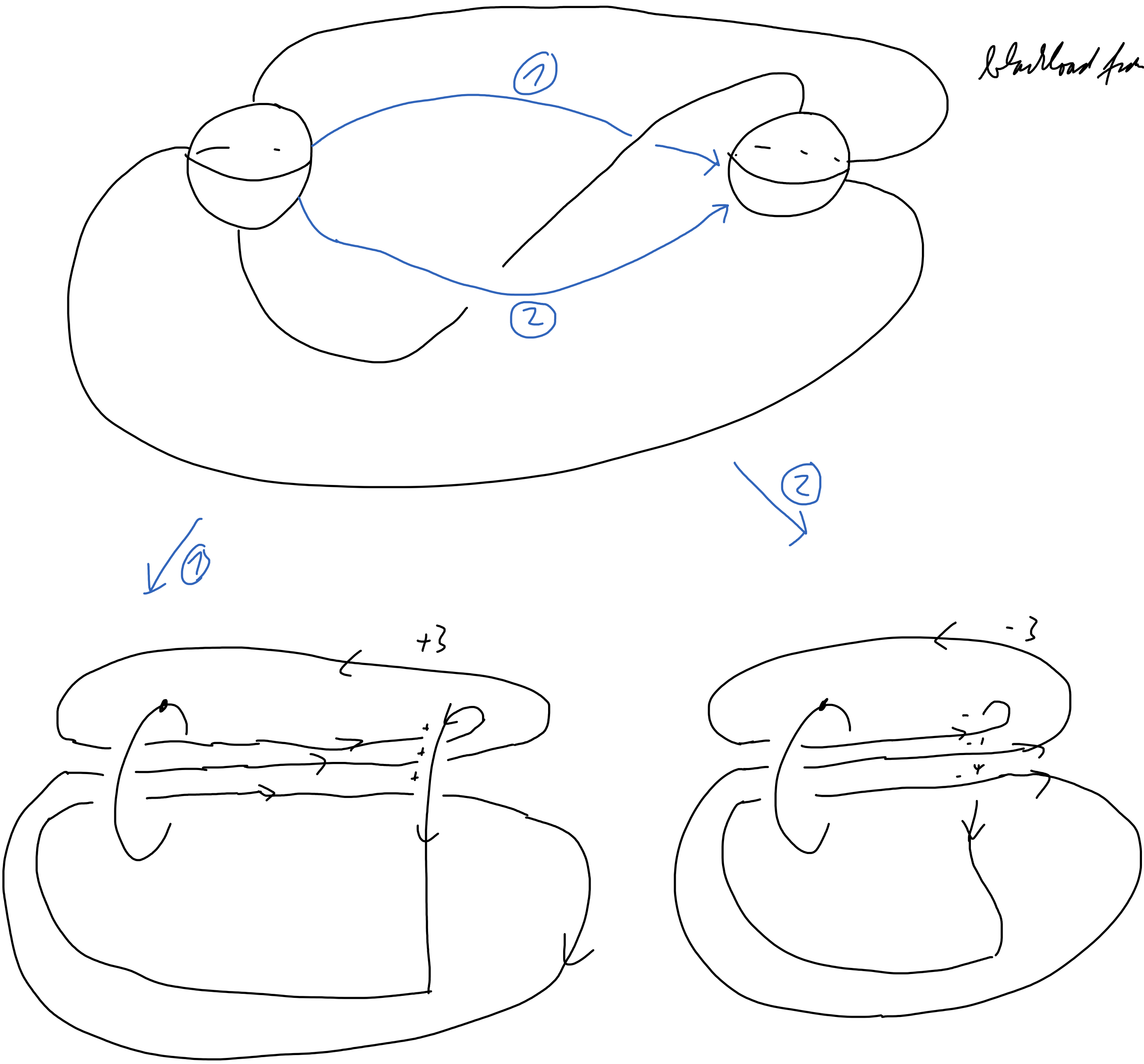
DRAW: unknot $K \subset S^3$ DOTTED

1-handle:



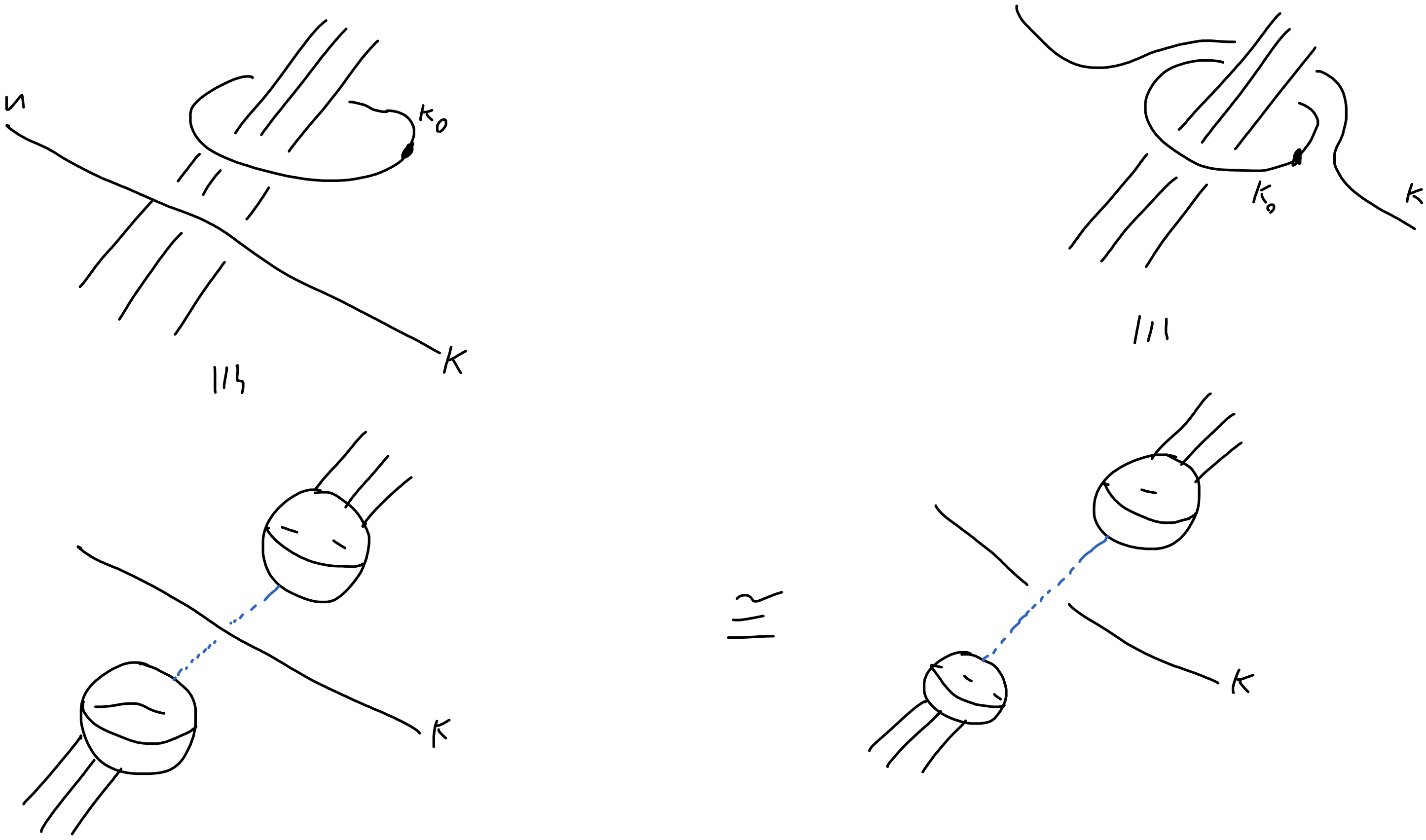
using dotted 1-handle

Ex:

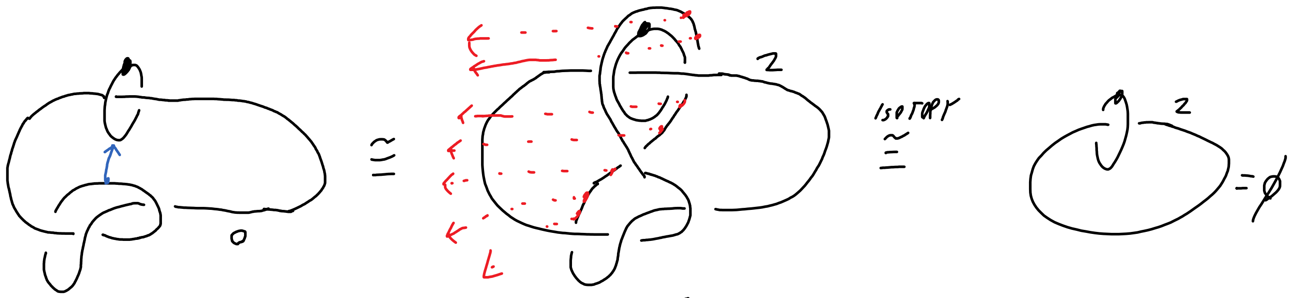


blackboard framing

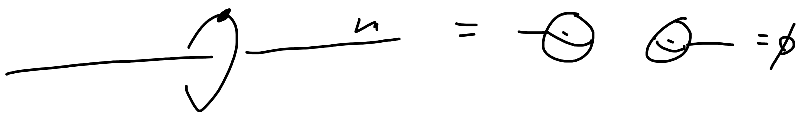
„2-HANDLE SLIDE UNDER A 1-HANDLE"



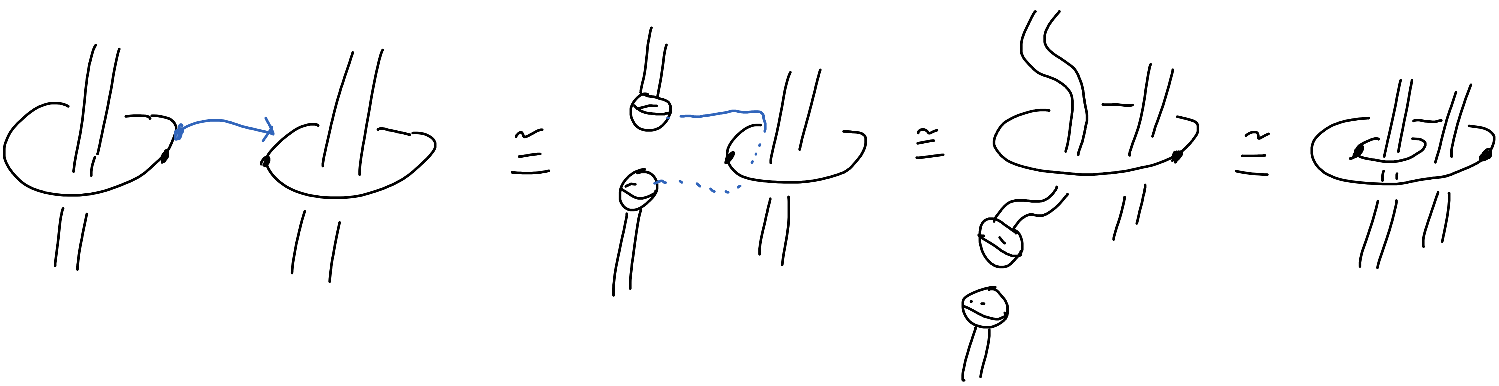
Ex:



CANCELLING 1-/2-HANDLE PAIR:

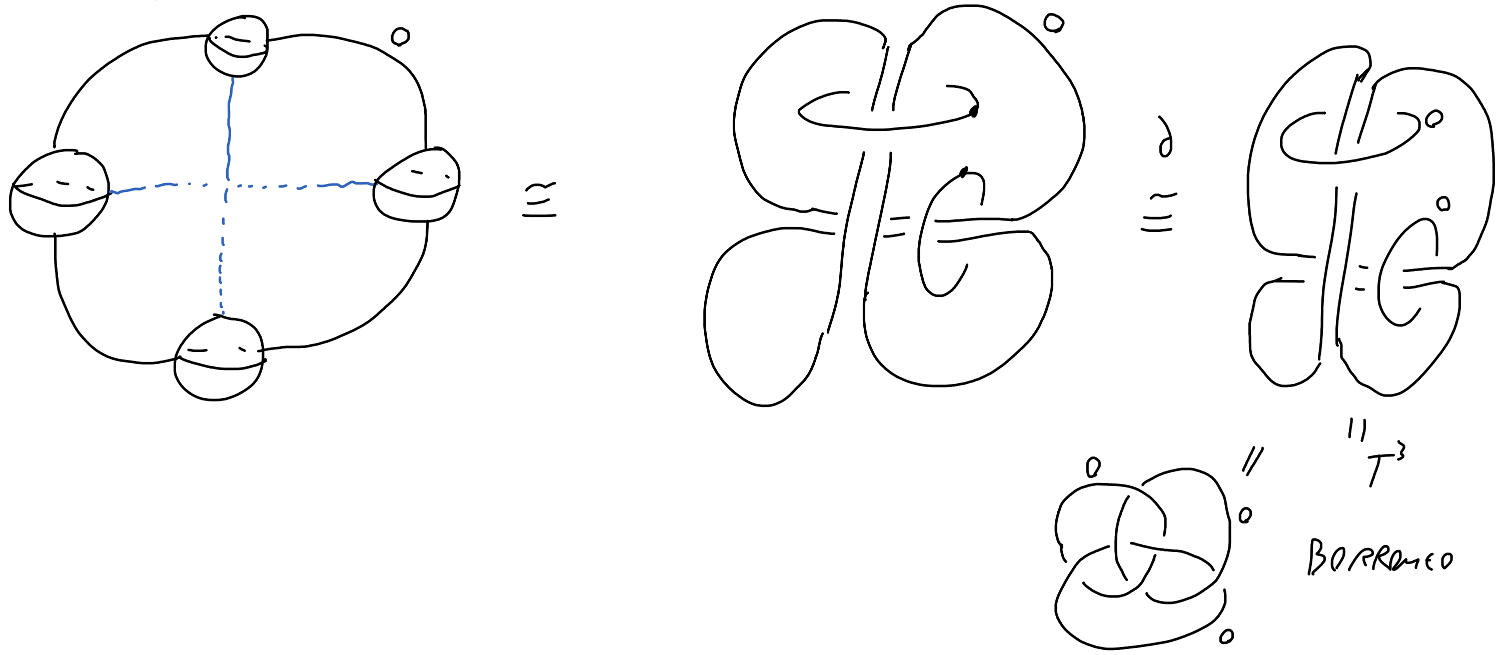


1-HANDLE SLIDE:

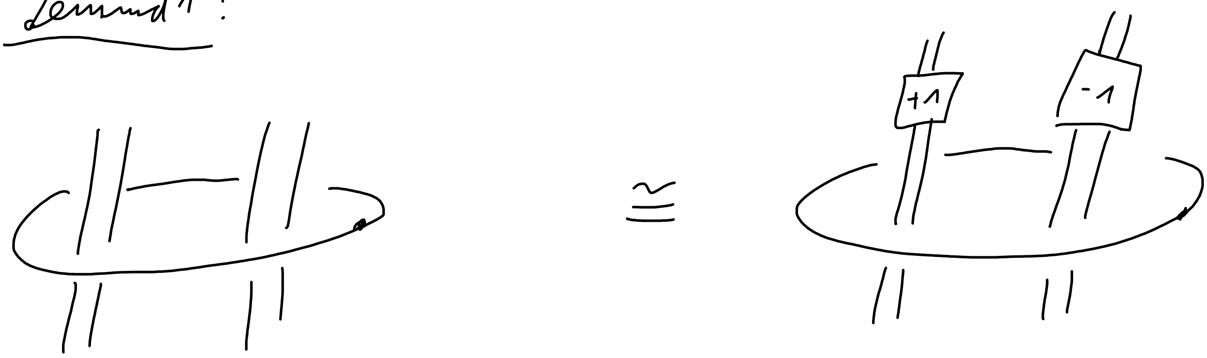


WARNING:  always needs to be an UNLINK!

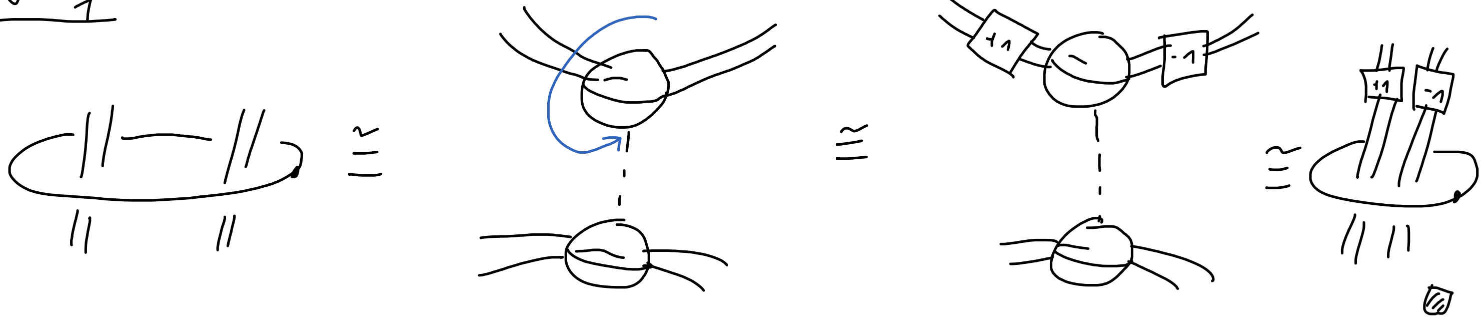
Ex: $T^2 \times D^2$



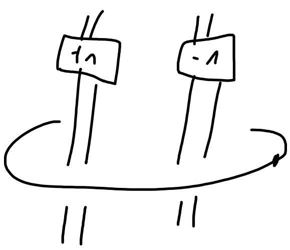
Lemma 1:



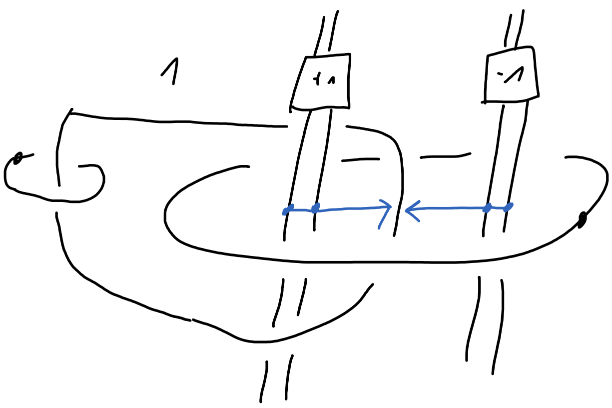
1. Proof:

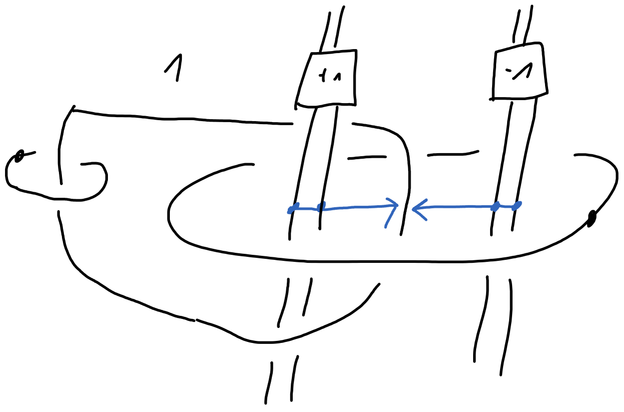


2. Proof:

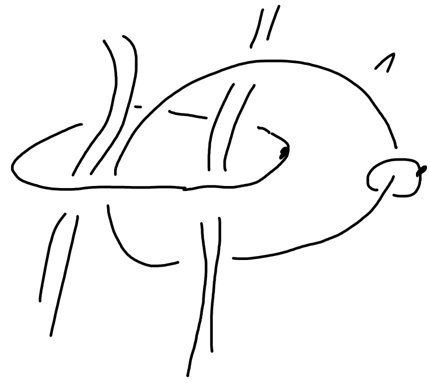


CANCELLING PAIR
≅



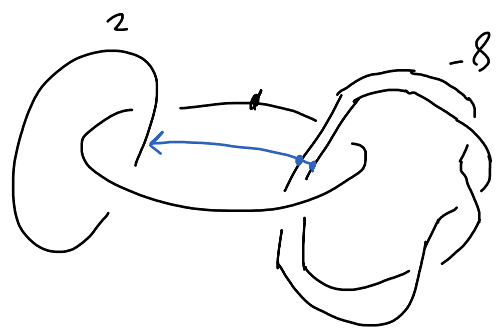


2-H.S.
$\cong$



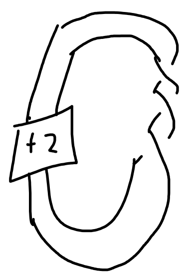
CANCEL
$\cong$

Ex:



SLIDE + CANCEL
$\cong$

$n_1 + \ell^2 n_2 + 2\ell\ell$
$\|$
$-8 + 4 \cdot 2 = 0$



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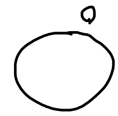


|| CANCEL

S^4

THE AKBULUT KIRBY SPHERE

FAKE $\mathbb{RP}^4$'s:

$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ induces a diffeo of $T^3 \setminus \mathring{D}^3$

$$E^4 = \left(\left(T^3 \setminus \mathring{D}^3 \times I \right) \Big/ _{(p,0) \sim (A(p),1)} \right) \cup S^2 \times D^2$$

via non-trivial diffeo of $S^2 \times S^1$

THM (CAPPELL-SHANESON '76)

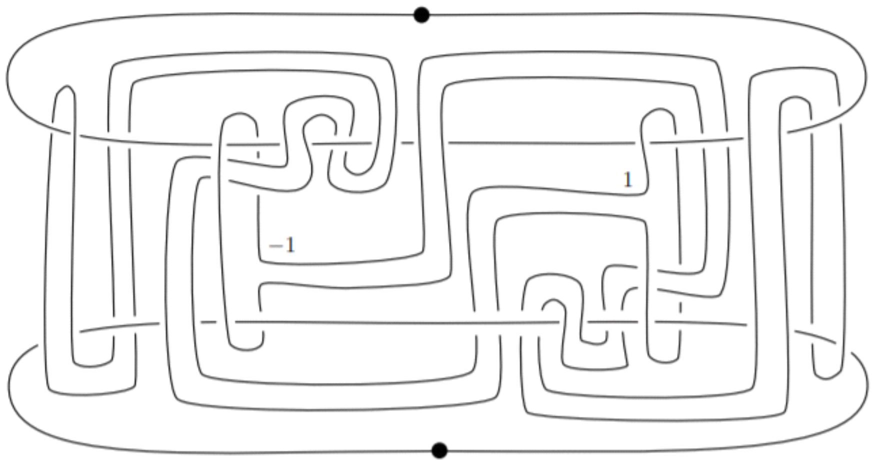
$E \simeq \mathbb{RP}^4$ but $E \overset{C^\infty}{\not\cong} \mathbb{RP}^4$

Proof: S. Cappell and J. Shaneson: Some New four-manifolds, https://www.jstor.org/stable/1971056?origin=crossref&seq=1#metadata_info_tab_contents ▨

COR: W = 2-fold cover of E $\simeq S^4$ and thus by Freedman thm $E \overset{C^0}{\cong} S^4$

$\boxed{\text{Q} \quad W \overset{C^\infty}{\cong} S^4 \ ?}$

THM (AKBULUT-KIRBY '85)



is a Kirby diagram of W

Proof: S. Akbulut and R. Kirby: A potential smooth counterexample in dimension 4 to the Poincare conjecture, the Schoenflies conjecture, and the Andrews-Curtis conjecture, https://www.sciencedirect.com/science/article/pii/0040938385900102?via%3Dihub ▨

THM (GOMPF '91)

$W \overset{C^\infty}{\cong} S^4$

Proof: perform Kirby calculus see sheet 7 ▨

R. Gompf: Killing the Akbulut-Kirby 4-sphere, with relevance to the Andrews-Curtis and Schoenflies problems, https://www.sciencedirect.com/science/article/pii/0040938391900364?via%3Dihub

9. COBORDISMS

9.1. THE COBORDISM RING

Def: Let M^n, N^n be smooth, closed (oriented) n-mfds

A (ORIENTED) COBORDISM is W^{n+1} smooth, compact (oriented)

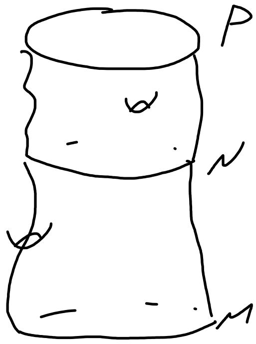
with $\partial W = M \sqcup (-)N$

$M \sim N :\Leftrightarrow M$ cobordant to N

Examples:

* $W = M \times [0,1)$ NOT a cobordism
* $S^1 \sim S^1 \sqcup S^1$

* $M \sim M$ via $W = M \times I$
* $M \sim N$ & $N \sim P \Rightarrow M \sim P$
* $M \sim N \Rightarrow N \sim M$

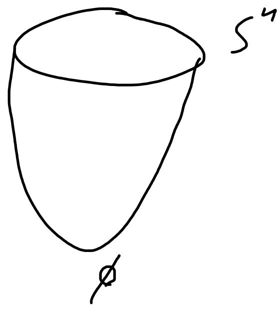


$\Rightarrow$ $\sim$ is an equivalence relation

$\mathcal{M}_n := \{n\text{-mfds}\}/_{\text{cob.}}$ $\qquad$ $\Omega_n^{SO} := \{\text{oriented } n\text{-mfds}\}/_{\text{or. cob.}}$

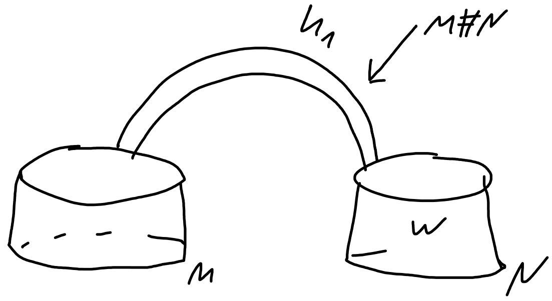
* $S^n \sim \emptyset$, i.e. NULL COBORDANT (FILLABLE)

$S^n = \partial D^{n+1}$



* $\Sigma_g \sim \emptyset$ $\left[\Sigma_g = \partial H_g \right]$

* $M \# N \sim M \sqcup N$



* $M \sim M'$ = surgery of M

Corollary 1:

$\mathcal{M}_*$ & Ω_*^{SO} carry the structure of a graded ring with

$+ = \#$ or $\sqcup$

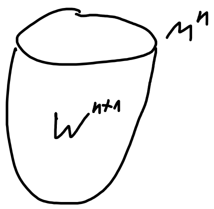
$\cdot = \times$

grading = dim

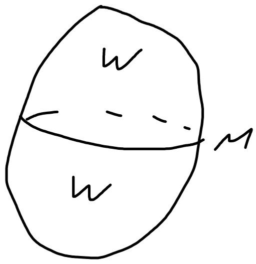
Ex: $* \mathbb{RP}^2 \not\sim \emptyset$, $\chi(\mathbb{RP}^2) = 1$

CLAIM: $M \sim \emptyset \Rightarrow \chi(M) \equiv 0 \mod(2)$

Proof:



$DW =$



$$\chi(DW) = 2\chi(W) - \chi(M)$$

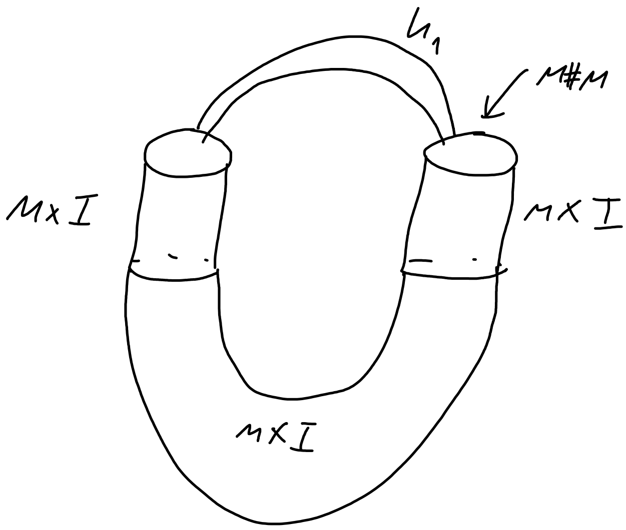
$$\chi(W) = \sum_{i=0}^{n+1} (-1)^i \# (h_i)$$

$$\chi(DW) = \sum_{i=0}^{n+1} (-1)^i \# (h_i) + \sum_{i=1}^{n+1} (-1)^{n+1-i} \# (h_i)$$

$$= \chi(W) - (-1)^{\dim(M)} \chi(W)$$

$$\chi(M) = \left(1 + (-1)^{\dim(M)}\right) \chi(W) \equiv 0 \mod(2)$$ ▨

$* M \# M \sim \emptyset$ (unoriented)



THM2:

$\mathcal{M}_0 = \mathbb{Z}$

$\mathcal{M}_1 = 0$

$\mathcal{M}_2 = \mathbb{Z}_2$ (gen. by $\mathbb{RP}^2$)

$\mathcal{M}_3 = 0$

$\mathcal{M}_4 = \mathbb{Z}_2^2$

$\mathcal{M}_5 = \mathbb{Z}_2$

$\vdots$

THM 3:

$\Omega_0^{SO} = \mathbb{Z}$ $\quad (\bullet\!\!-\!\!\bullet)$

$\Omega_1^{SO} = 0$ $\quad (S^1 = \partial D^2)$

$\Omega_2^{SO} = 0$ $\quad (\Sigma_g = \partial H_g)$

$\Omega_3^{SO} = 0$ $\quad (M^3 = \partial W^4)$

$\Omega_4^{SO} = \mathbb{Z}$ $\quad \tau: \Omega_4^{SO} \xrightarrow{\cong} \mathbb{Z}$ SIGNATURE

$\Omega_5^{SO} = \mathbb{Z}_2$

For more on cobordisms see for example:
https://en.wikipedia.org/wiki/Cobordism

$\Omega_6 = \Omega_7 = 0$

$\Omega_{n\geq 8} \neq 0$

9.2. THE h-COBORDISM THM & THE POINCARÉ CONJECTURE

Def: An oriented cob. W with $\partial W = M_1 \sqcup -M_0$ is called h-COBORDISM :$\Leftrightarrow$

(1) $\pi_1(M_i) = 1 = \pi_1(W)$

(2) $H_*(M_i) \xhookrightarrow{i_*} H_*(W)$ are isom or

$H_*(W, M_i) = 0$

Ex: $W = M \times I$ (with $\pi_1(M) = 1$)

THM 4 (SMALE)

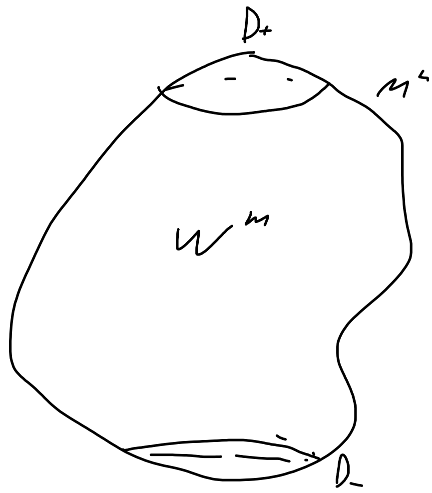
Let W be an h-cob. between M_0 & M_1 of $\dim(W) > 5$

$\Rightarrow W \overset{C^\infty}{\cong} M_0 \times I \overset{C^\infty}{\cong} M_1 \times I$

Corollary 5:

Let M^n with $n \geq 6$ & $M^n \simeq S^n$

$\Rightarrow M \overset{C^0}{\cong} S^n$



Proof: $W^n := M^n \setminus (\mathring{D}^n_+ \sqcup \mathring{D}^n_-)$

$\Rightarrow W^n$ is an h-cob from S^{n-1} to S^{n-1}

$\overset{\text{THM 4}}{\Rightarrow} W^n \overset{C^\infty}{\cong} S^{n-1} \times I$

$\Rightarrow M^n \overset{C^\infty}{\cong} D^n_- \cup S^{n-1} \times I \cup D^n_+ \overset{C^0}{\cong} S^n$

ALEXANDER TRICK

$\blacksquare$

Remark: * THM 4 is also true for top 4-mfds (FREEDMAN)

* But is WRONG for smooth 4-mfds.

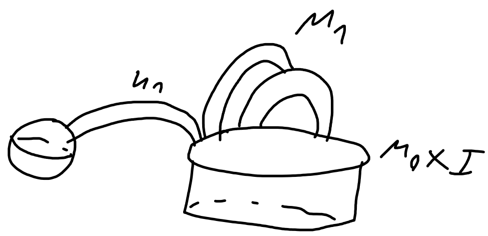
Proof of THM 4:

We show by induction after k:

$\exists$ handl. decomp of W^n without i-handles for $i < k$.

$\boxed{\text{BASE}}$ $\boxed{k=1}$ cancel all 0-handles

$\boxed{k=2}$ handle trade (later)



$\boxed{\text{IS}}$ Assume: that W has a handl. decomp without i-handles for $i < k$

GOAL: $\exists$ handl. decomp. of W without h_i for $i < k+1$

Let h_k be a k-handle

$\Rightarrow h_k \in C_k(W, M_0)$

$H_k(W,M_0) = 0$ & $C_{k-1}(W, M_0) = 0$

$\Rightarrow \exists\, e \in C_{k+1}(W, M_0)$ s.t. $\partial e = h_k$

$\Rightarrow e = \sum c_i h^i_{k+1} \in C_{k+1}(W, M_0)$

After a basis transformation ($\hat{=}$ $(k+1)$-handle slides), we can assume that $e = h_{k+1}$ & $\partial h_{k+1} = h_k$

$\Rightarrow a(h_{k+1}) \cdot b(h_k) = 1$

If $a(h_{k+1}) \pitchfork b(h_k) = \{pt\}$ we can cancel h_k & h_{k+1}

The claim follows from the following lemma:

<u>Lemma 6</u> (WHITNEY TRICK)

Let N_1^n & $N_2^l \overset{C^\infty}{\subset} Y^m$ with $n + l = m > 4$

$N_1 \cdot N_2 = \pm 1$ & $\pi_1(Y \setminus N_1 \cup N_2) = 1$

$\Rightarrow N_1$ is isotopic to N_1' with $N_1' \pitchfork N_2 = \{pt\}$

Since $k \geq 3$ we have:

$N_1 = a(h_{k+1})$, $\dim(N_1) = k \geq 3$, $Y = \partial(M_0 \times I \cup h^i_k \cup h_{k+1})$

$N_2 = b(h_k)$, $\dim(N_2) = n-k-1 \leq n-4$, $\dim(Y) = n-1 > 4$

$\Rightarrow \pi_1(Y \setminus (N_1 \cup N_2)) = 1$

Proof detail of L.6

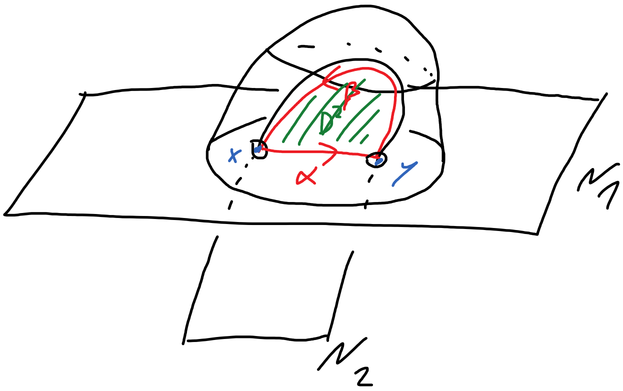
Let $\#(N_1 \pitchfork N_2) > 1$

$\Rightarrow$ $\exists$ $x, y \in N_1 \pitchfork N_2$ of opposite signs

choose α = path from x to y in N_1

β = " y to x in N_2

$\Rightarrow$ $\gamma = \alpha \cdot \beta$ is a loop isotopic to a loop in $Y \setminus (N_1 \cup N_2)$ (TO CHECK!)



$\pi_1(Y \setminus (N_1 \cup N_2)) = 1$

$\Rightarrow$ γ bounds an immersed disk D^2 in $Y \setminus (N_1 \cup N_2)$

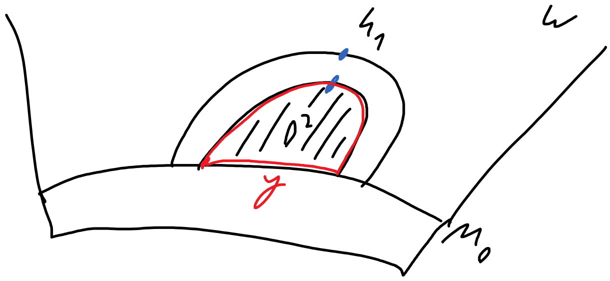
$n > 4$
$\Rightarrow$ D^2 can be perturbed to be embedded

$\Rightarrow$ isotope N_1 „over" D^2 to N_1' $\blacksquare$

HANDLE TRADE: $\boxed{k=2}$

Let h_1 be a 1-handle.

$\gamma \in \pi_1(W) = 1$



$\Rightarrow$ γ bounds an immersed disk D^2 in W

$\dim(W) > 5$
$\Rightarrow$ D^2 can be perturbed to be embedded

$\Rightarrow$ thicken D^2 to a cancelling 2-/3-handle pair:

$$\cup D^2 = \underbrace{D^2 \times D^{n-2}}_{h_2} \cup \underbrace{D^3 \times D^{n-3}}_{h_3} \quad \text{s.t.}$$

h_1 & h_2 cancel each other

h_2 & h_3 " $\blacksquare$

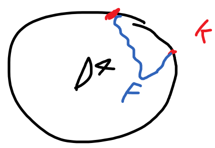
10. EXOTIC 4-MANIFOLDS:

10.1. SLICE KNOTS:

Let K be an oriented knot in S^3

GENUS: $g(K) = \min\{g(F) \mid F \subset S^3 \text{ compact, oriented s.t. } \partial F \overset{\text{a.}}{=} K\}$

* $g(K) = 0 \Leftrightarrow K = \text{unknot}$.



SMOOTH 4-GENUS:

$g_{C^\infty}(K) := \min\{g(F) \mid F \overset{C^\infty}{\subset} D^4 \text{ comp. oriented s.t. } \partial F = K\}$

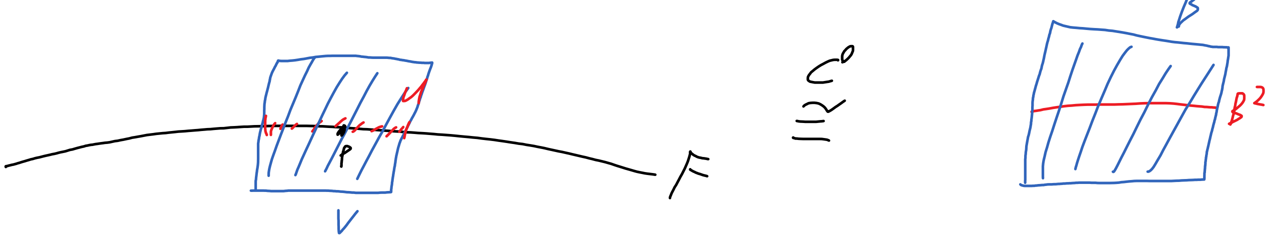
TOPOLOGICAL 4-GENUS:

$g_{C^0}(K) := \min\{g(F) \mid F \overset{\text{locally flat}}{\subset} D^4 \quad \text{"} \quad \}$

$F \subset D^4$ is LOCALLY FLAT $:\Leftrightarrow$

$\forall p \in F \ \exists$ nbhd U of p in F & nbhd V of p in D^4 s.t.

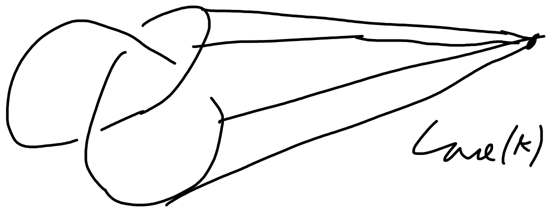
$$(U, V) \overset{C^0}{\cong} (B^2, B^4)$$



K is called (TOP) SLICE knot $:\Leftrightarrow$ $g_{C^\infty}(K) = 0$ $\left(g_{C^0}(K) = 0\right)$

Remark: we need the local flatness:

$$\mathrm{Cone}(K) \subset D^4 = \mathrm{Cone}(S^3) \overset{C^0}{\cong} D^2 \subset D^4$$

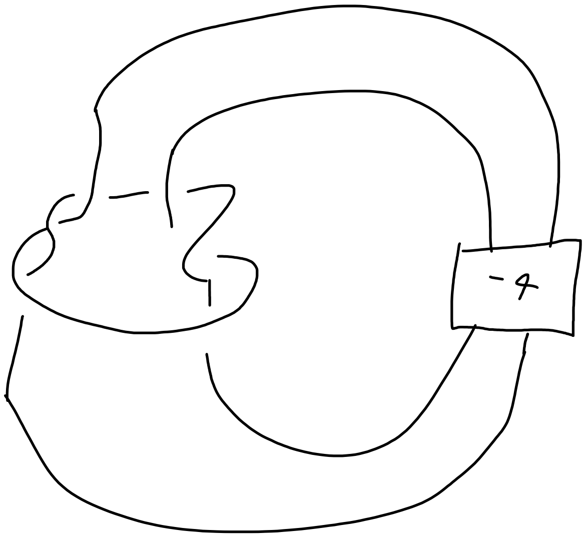


but NOT locally flat

Corollary 1: $g_{C^0}(K) \le g_{C^\infty}(K) \le g(K)$ ∎

Example 2:

$K =$

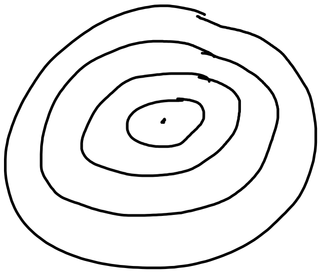


$\neq O$

$\Rightarrow g(K) \neq 0$

CLAIM: $g_{C^\infty}(K) = 0$ (i.e. $g_{C^\infty} < g$)

Proof: $D^4 = \bigcup_{r \in [0,1]} S^3_r$



$r = 1$

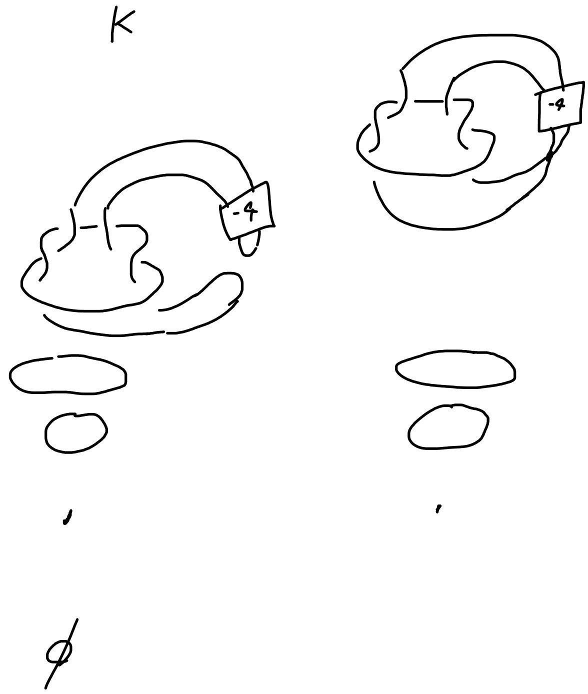
$r = 1/2$

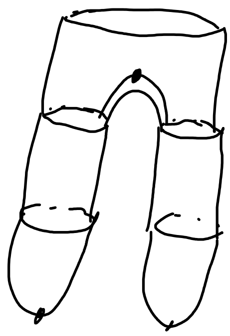
$r = 1/3$

$r = 1/4$

$r = 1/5$

$r = 1/6$





Remark: Knots which can be constructed as in Ex 2 are called RIBBON.

CONJECTURE: $g_{C^\infty}(K) = 0 \overset{?}{\Leftrightarrow} K$ is Ribbon

Corollary 3 $g_{C^\infty}(K \# -K) = 0$

$-K$ = mirror of K with opposite or.

CONCORDANCE GROUP

$C := \{$ or. knots in $S^3\} / \sim$

where $K_1 \sim K_2 :\Leftrightarrow g_{C^\infty}(K_1 \# -K_2) = 0$

is an abelian group with op $\#$

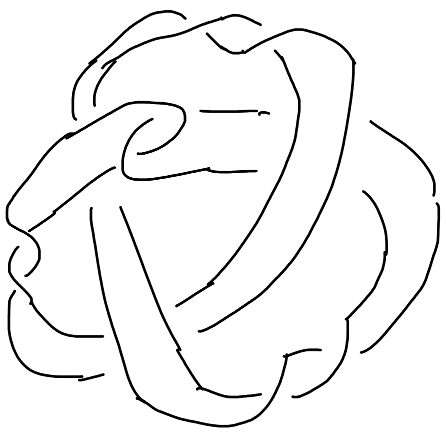
THM 4 (FREEDMAN)

Let Δ_K be the ALEXANDER POLYNOMIAL of K.

$$\left(\text{def by } \Delta_{\nwarrow\!\!\!\!\nearrow} - \Delta_{\nwarrow\!\!\!\!\nearrow} + \left(t^{-1/2} - t^{1/2}\right)\Delta_{\uparrow\uparrow} = 0 \quad \& \quad \Delta_{\circ} = 1 \right)$$

$$\Delta_K(t) = 1 \Rightarrow g_{C^0}(K) = 0 \qquad \blacksquare$$

Examples: W_+R = POSITIVE WHITEHEAD DOUBLE OF THE RIGHT HANDED TREFOIL

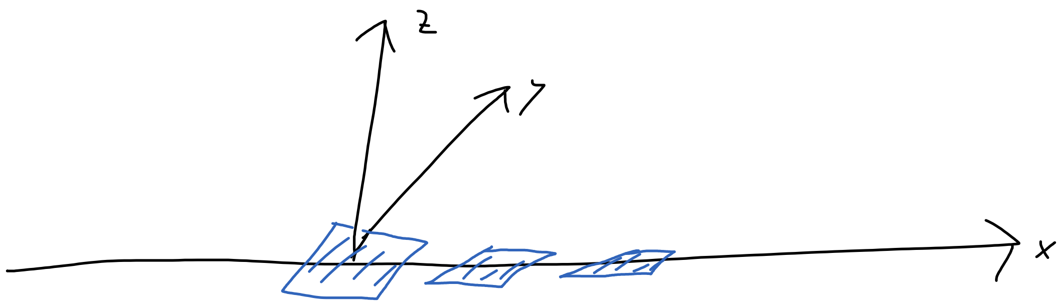


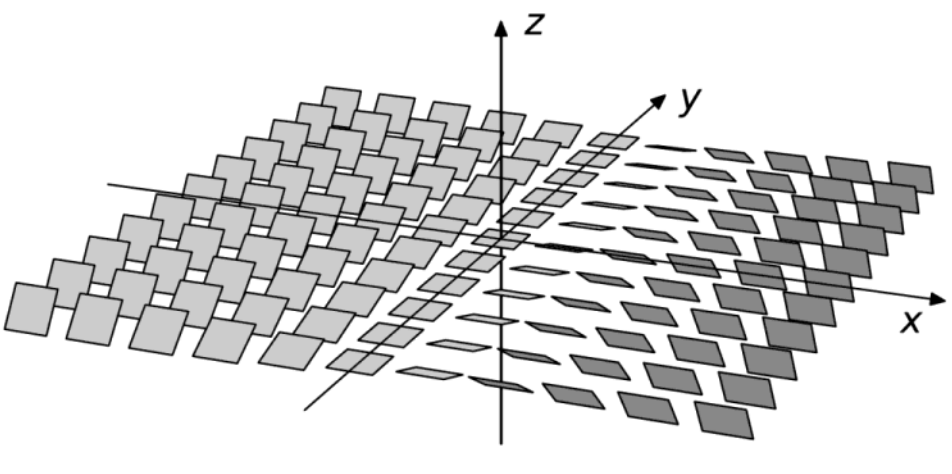
$$\Delta_{W_+R}(t) = 1$$

$$\Rightarrow g_{C^0}(W_+R) = 0$$

10.2. THE BENNEQUIN-BOUND:

Ex: $\xi_{st} = \langle \partial_x, \partial_y - x\partial_z \rangle$ is called STANDARD CONTACT STRUCTURE on $\mathbb{R}^3_{(x,y,z)}$





The standard contact structure. This figure is (except for some small changes in colors and axes) retrieved from Wikipedia created by user Msr657 available online at https://en.wikipedia.org/wiki/File:Standard_contact_structure.svg

$$\xi_{st} = \ker(x\,dy + dz)$$

Def: $\ker(\alpha) = \xi^2 \subset TM$ is CONTACT $:\Leftrightarrow$ $\alpha \wedge d\alpha$ is a volume form

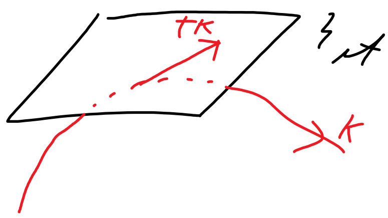
Ex: $S^3 \subset \mathbb{R}^4$, $\xi_{st} = \ker\left(\sum_{i=1}^{2} x_i\,dy_i - y_i\,dx_i \right)$

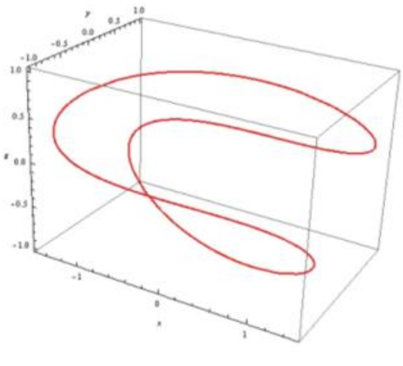
Facts: * $(S^3 \setminus pt, \xi_{st}) \overset{cont}{\cong} (\mathbb{R}^3, \xi_{st})$

* $\nexists$ surface $F \subset (M, \xi)$ s.t. $TF = \xi|_F$

Def: $K \subset (\mathbb{R}^3, \xi_{st}) \subset (S^3, \xi_{st})$ is called LEGENDRIAN

$:\Leftrightarrow$ $TK \subset \xi_{st}$





$[0, 2\pi] \ni t \longmapsto \left(x(t) = 3\sin(t)\cos(t), y(t) = \cos(t), z(t) = \sin^3(t)\right) \in \mathbb{R}^3.$

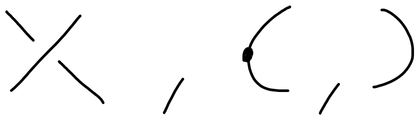
Ex: FRONT PROJECTION:

$$(x,y,z) \longmapsto (y,z) \qquad \xi_{st} = \ker(x\,dy + dz)$$

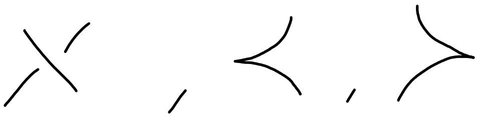
$K(t) = (x(t), y(t), z(t))$ Leg $\Leftrightarrow$ $x(t)\,y'(t) + z'(t) = 0$

i.e. we get K from its front projection

forbidden:

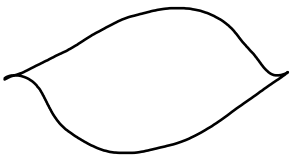


allowed:

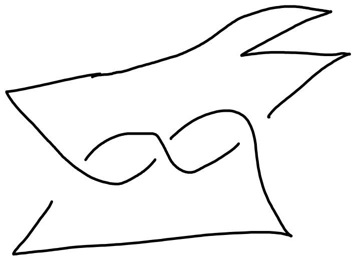


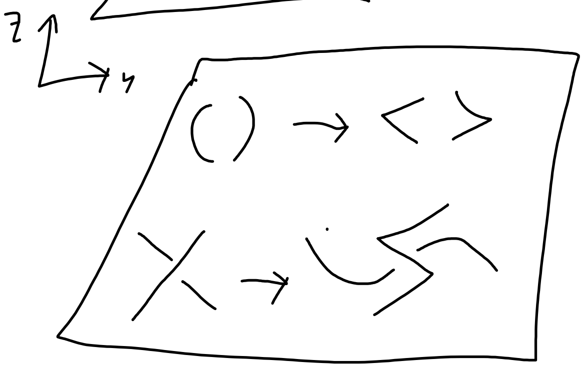
Ex:

U

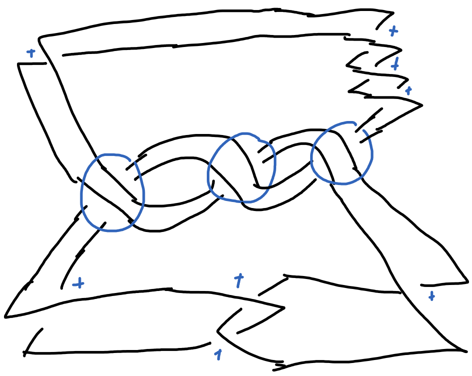


$R =$





$W_+R =$



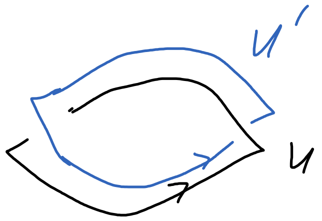
Def: THURSTON-BENNEQUIN INVARIANT in (S^3, ξ_{st})

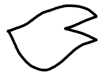
$$tb(K) := lk(K, K')$$

⋔ to ξ_{st}

where K' is the push-off of K in z-direction.

Ex:





$tb(u) = lk(u, u') = -1$

Corollary 6: $tb(K) = -\frac{1}{2}\#\text{cusps} + w$ (in the front)

$\uparrow$ writhe

Ex: $tb(W_+R) = -1/2\,(14) + 8 = 1$

THM 7: (BENNEQUIN BOUND, RUDOLPH)

$$tb(K) \leq 2 \cdot g_{C^\infty}(K) - 1$$

Proof (maybe later) $\square$

Example 8: $1 = tb(W_+R) \leq 2\, g_{C^\infty}(W_+R) - 1$

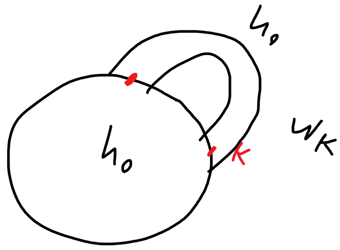
$\Rightarrow \quad g_{C^\infty}(W_+R) \geq 1$

$\Rightarrow \quad \boxed{\text{i.e. } g_{C^0} < g_{C^\infty}}$

10.3. Exotic $\mathbb{R}^4$'s

Let $K \subset \partial D^4$ be a knot

$W_K := h_0 \cup h_2$ attached along K with framing 0



Lemma 9:

$g_{C^0}(K) = 0 \iff \exists f : W_K \xhookrightarrow{C^0} \mathbb{R}^4$

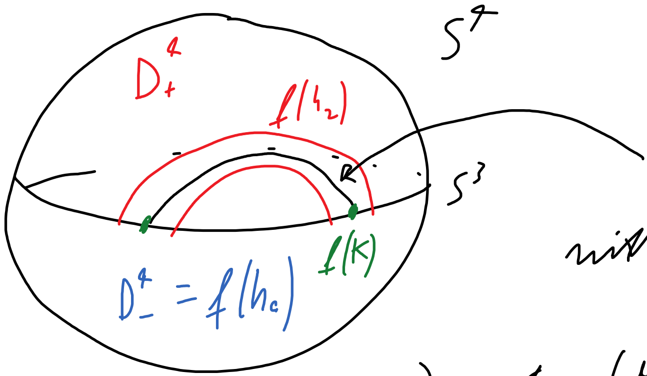
Lemma 10:

$g_{C^\infty}(K) = 0 \iff \exists f : W_K \xhookrightarrow{C^\infty} \mathbb{R}^4$

Proof of L10: „$\Leftarrow$"

Let $f : W_K \xhookrightarrow{C^\infty} \mathbb{R}^4 \subset S^4$

$\Rightarrow f(h_0) \overset{C^\infty}{\cong} D^4_- \subset S^4$ & $S^4 \setminus f(h_0) \overset{C^\infty}{\cong} D^4_+$



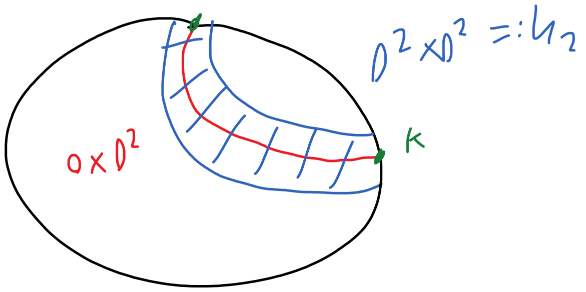
core of $h_2 = D^2 \subset D^4_+$

with $\partial D^2 = K \subset S^3 = \partial D^4_+$

$\Rightarrow g_{C^\infty}(K) = 0$ $\blacksquare$

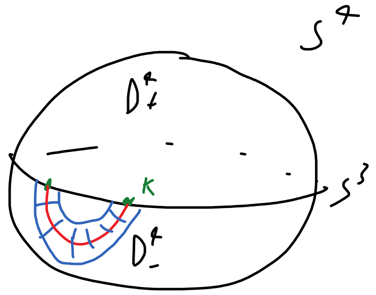
Proof of L.9. „$\Rightarrow$)"

$$g_{C^0}(K) = 0 \Rightarrow \exists\ D^2\times D^2 \xhookrightarrow{C^0} D^4_- \quad \text{s.t.}$$



$D^4_- = h_0$

write $S^4 = D^4_+ \cup D^4_-$



$$D^4_+ \cup D^2\times D^2 = h_0 \cup h_2 = W_K$$

$$\Rightarrow f: W_K \xhookrightarrow{C^0} S^4$$

W_K is compact

$$\Rightarrow \exists f: W_K \xhookrightarrow{C^0} \mathbb{R}^4 \qquad \blacksquare$$

THM 11:

$\exists$ a smooth 4-mfd R^4 s.t. $R^4 \overset{C^0}{\cong} \mathbb{R}^4$ but $R^4 \overset{C^\infty}{\ncong} \mathbb{R}^4$

Proof: Let $K = W_+ R$ $\quad$ ($g_{C^0}(K) = 0$ but $g_{C^\infty}(K) \neq 0$)

$$\overset{L.9.}{\Rightarrow} \exists f: W_K \xhookrightarrow{C^0} \mathbb{R}^4$$

$\Rightarrow \mathbb{R}^4 \setminus f(W_K)$ is a smooth 4-mfd & $\partial(\mathbb{R}^4\setminus f(W_K)) \underset{\varphi}{\overset{C^0}{\cong}} \partial W_K$

Moise

$\Rightarrow \exists g: \partial(\mathbb{R}^4\setminus f(W_K)) \xrightarrow[\cong]{C^\infty} \partial W_K$ isotopic to φ

$R := \mathbb{R}^4 \setminus f(W_K) \cup_g W_K$ is a smooth 4-mfd with $R^4 \overset{C^0}{\cong} \mathbb{R}^4$

But L10 $\Rightarrow R^4 \ncong \mathbb{R}^4$, since $W_K \xhookrightarrow{C^\infty} R^4$ $\qquad \blacksquare$

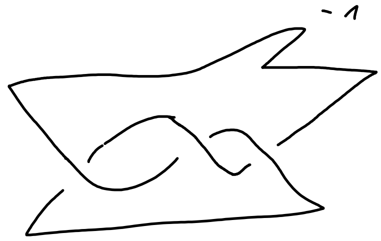
10.4. THE ADJUNCTION INEQUALITY

<u>Def</u>: W^4 is <u>STEIN</u> $:\Leftrightarrow$ $W^4 = h_0 \cup \{1\text{-handles}\} \cup \{2\text{-handles}\}$

where the 2-handles are attached along Legendrian knots

with framing tb − 1

<u>Ex</u>: D^4 is Stein,



is Stein

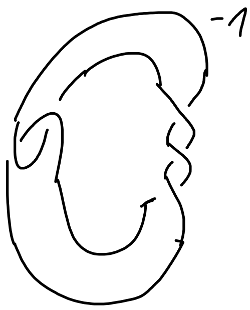
<u>THM 12</u> (ADJUNCTION INEQUALITY)

W^4 Stein, $\Sigma^2 \subset W^4$ compact, smooth, oriented

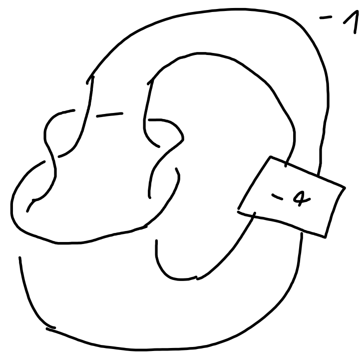
$$\Rightarrow \quad \Sigma \cdot \Sigma \leq 2g(\Sigma) - 2$$

(exception: $\Sigma = S^2$ & $[\Sigma] = 0 \in H_2(W)$) $\blacksquare$

$W_1 =$

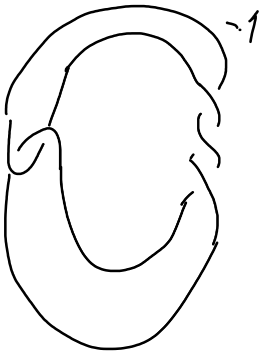


$W_2 =$

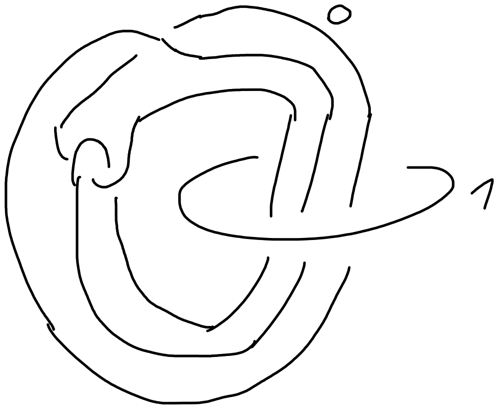


<u>THM 13</u> $W_1 \overset{C^0}{\cong} W_2$ but $W_1 \overset{C^\infty}{\not\cong} W_2$

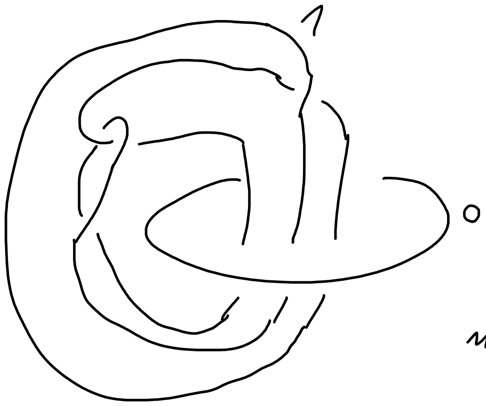
Proof: ① $\partial W_1 \overset{C^\infty}{\cong} \partial W_2$



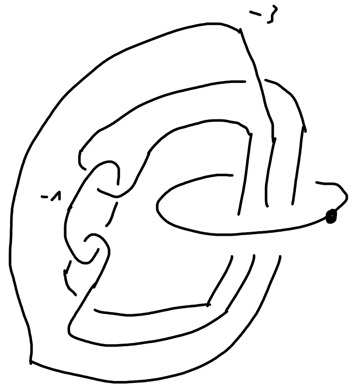
$\overset{\partial}{\cong}$ blowup



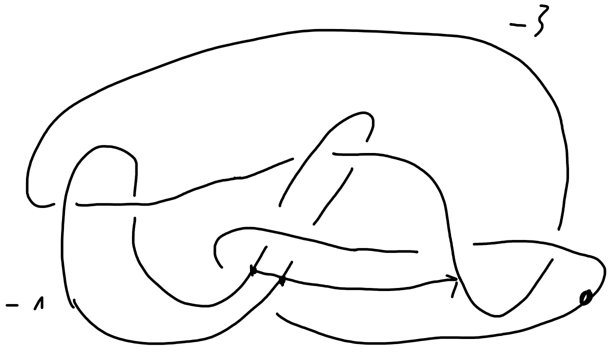
isotopy $\cong$



blowup $\overset{\partial}{\cong}$ & surgery



isotopy $\cong$



2-handleslides & cancellation $\cong$

② $Q_{W_1} = Q_{W_2} = (-1)$

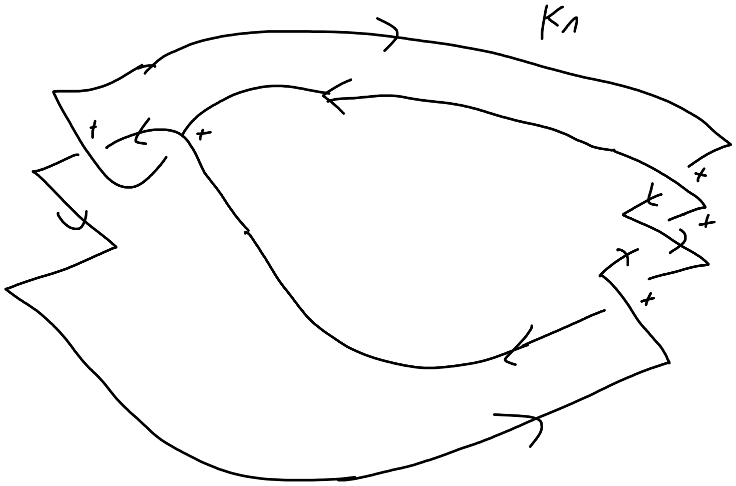
$\Rightarrow H_*(\partial W_i) = H_*(S^3)$

& $\pi_1(W_i) = 1$

(for 4-mfds with $\partial W_1 \cong \partial W_2$ homology S^3's)

Freedman
$\Rightarrow W_1 \overset{C^0}{\cong} W_2$

③ W_1 carries a Stein structure:



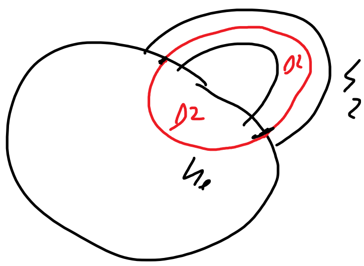
with $tb(K_1)$ =

$-\frac{1}{2}(10) + 5 = 0$

④ W_2 carries <u>no</u> Stein structure.

$H_2(W_2) = \langle F = S^2 \rangle_{\mathbb{Z}}$

$S^2 \cong F$ = core of h_2 $\cup$ slice disk of K_2 in h_0 (Ex 2)



$Q = (-1) \Rightarrow F \cdot F = -1$

& $2g(F) - 2 = -2$

THM 13 $\Rightarrow$ W_2 carries <u>no</u> Stein structure. ∎

<u>Remark</u>: $DW_1 \overset{C^\infty}{\cong} DW_2$ by C.S.G.

For a combinatorial proof of the slice Bennequin bound see:
J. Rasmussen: Khovanov homology and the slice genus,
https://link.springer.com/article/10.1007%2Fs00222-010-0275-6

For the original proof see:
L.Rudolph: Quasipositivity as an obstruction to sliceness,
https://www.ams.org/journals/bull/1993-29-01/S0273-0979-1993-00397-5/home.html
and
L. Rudolph: An obstruction to sliceness via contact geometry and "classical'' gauge theory,
https://link.springer.com/article/10.1007%2FBF01245177

For a gauge-theory-free proof of the adjunction inequality see:
P. Lambert-Cole: Symplectic trisections and the adjunction inequality,
https://arxiv.org/pdf/2009.11263.pdf

For the original proof see:
P. Kronheimer and T. Mrowka: The genus of embedded surfaces in the projective plane,
https://www.intlpress.com/site/pub/pages/journals/items/mrl/content/vols/0001/0006/a014/
J. Morgan, Z. Szabó, and C. Taubes: A product formula for the Seiberg-Witten invariants and the generalized Thom conjecture,
https://projecteuclid.org/journals/journal-of-differential-geometry/volume-44/issue-4/A-product-formula-for-the-Seiberg-Witten-invariants-and-the/10.4310/jdg/1214459408.full